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M.Sc. MATHEMATICS

II YEAR

MECHANICS

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SMAM34: MECHANICS

SYLLABUS

Unit I: Introductory Concepts: The Mechanical system- Generalized coordinates – Constraints - Virtual work - Energy and Momentum

Chapter 1: Sections 1.1 to 1.5

Unit II: Lagrange's Equations: Derivation of Lagrange's Equations-Examples- Integrals of the motion.

Chapter 2: Sections 2.1 to 2.3

UNIT III: Hamilton's Equations: Hamilton's Principle - Hamilton's Equations - Other variation principles.

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Unit IV: Hamilton-Jacobi Theory: Hamilton’s Principle function – Hamilton-Jacobi Equation - Separability

Chapter 4: Sections 4.1 to 4.3

Unit V: Canonical Transformations: Differential forms and generating functions – Special Transformations– Lagrange and Poisson brackets.

Chapter 5: Sections 5.1 to 5.3

Text Book:

1. D. Greenwood, Classical Dynamics, Prentice Hall of India, New Delhi, 1985.

Reference Book

1. H. Goldstein, Classical Mechanics, (2nd Edition) Narosa Publishing House, New Delhi.

2. N.C. Rane and P.S.C. Joag, Classical Mechanics, Tata McGraw Hill, 1991.

3. J.L. Synge and B.A. Griffth, Principles of Mechanics (3rdEdition) McGraw Hill Book Co., New York, 1970

SMAM34: MECHANICS

CONTENTS

UNIT-I

Mechanical system- Generalized co-ordinates- Constraints-Virtual Work-Energy and momentum

Chapter 1: Sec 1.1 - 1.5

1.1.Mechanical System:

Let us consider a mechanical system consisting of N particles, where a particle is an idealized material body having its mass concentrated at a point. The motion of a particle is therefore the motion of a point in space. Since a point has no geometrical dimensions we cannot specify the orientation of a particle, nor can we associate any particular rotational motion with it.

Equations of motion:

The differential equations of motion for a system of N particles can be obtained by applying Newton's laws of motion to the particles individually. For a single particle of mass m which is subject to a force F we obtain from Newton's second law the vector equation.

Let $\vec{F}$ be the force acting on the particle and $\vec{P} = m\vec{v}$ (linear momentum)

∴ The equation of motion of the particle.

$$\vec{F} = m\vec{a}$$
$$= m\frac{d\vec{v}}{dt}$$
$$= \frac{d}{dt}(m\vec{v})$$
$$\vec{F} = \frac{d}{dt}(\vec{P})$$

is called equation of motion.

Let us consider a system of n particle Let $\vec{r}_i$ be the position vector of i^{th} particle of mass m_i.Then equation of motion are $m_i\ddot{r}_i = \vec{F}_i + \vec{R}_i, i = 1 \dots N$

where $\vec{F}_i$ is the applied force of i^{th} particle $\vec{R}_i$ is the constrain force.

Let each of the particle of the system we can assign a rectangular co-ordinate. (x_i, y_i, z_i)

There are three Cartesian co-ordinates

∴ The equation of motion is

$$m_i\ddot{x}_i = F_{ix} + R_{ix}$$
$$m_i\ddot{y}_i = F_{iy} + R_{iy}$$
$$m_i\ddot{z}_i = F_{iz} + R_{iz} \qquad \text{(i=1, 2, ………. N)}$$

1.2.Generalized coordinates:

1.Degree of Freedom:

The number of degrees of freedom of a system is equal to number of co-ordinates minus the number of independent equation of the constraint.

For example, if the configuration of a system of N particles is described using 3N Cartesian co-ordinates, and if there are l independent equation of constrain relating these co-ordinates, then there are $(3N - l)$ degrees of freedom.

Example 1:

Suppose that three particles are connected by a rigid rod to form triangular body with the particles at its corners. The configuration of the system is specified by giving the locations of the three particles, that is by 9 Cartesian coordinates. But each rigid rod is represented mathematically by an independent equation of constraint.

The no of degree freedom $= 3N - l$

$$= 3(3) - 3$$
$$= 9 - 3 = 6$$

And the system has six degrees of freedom.

2.Generalized Coordinates:

There are minimum no. of co-ordinate required to fix a configuration (position) of a system at any time t.They may be point, angle is time. They may also have been function of this three variables. This co-ordinate will be independent. When a if is possible to varied independently without violating geometrical constraints of the system. Once a set of independent co-ordinate for a system is known can be located uniquely at the time.

3.Transformation of Equation:

Let us consider the transformation of Equation relative to the Cartesian Co-ordinate

$x_1 \cdot x_2 \ldots x_{3N}$ to the Generalized co-ordinate $q_1, q_2 \ldots q_{\mathrm{n}}$

We assume this equation of the form

$$x_1 = x_1(q_1, q_2 \ldots . q_n, t)$$
$$x_2 = x_2(q_1, q_2 \ldots - q_{\mathrm{n}}, t)$$
$$\vdots$$
$$x_{3N} = x_{3N}(q_1, q_2 \ldots q_n, t)$$

If the x axis have l equation of constraints and q's hove N equation of constrain. Then No. of degree of freedom $= 3N - l$

This equation is may be solved for the $q's$ in terms of x and t. provided Jocibian of transformation of equation is non zero.

$$\therefore \frac{\partial(x_1 \dots \dots x_{3N})}{\partial(q_1 \cdot q_2 \dots q_{3N})} \neq 0$$

Example 1:

Consider the particle which the constrain moved on a fixed circular form part of radius a.

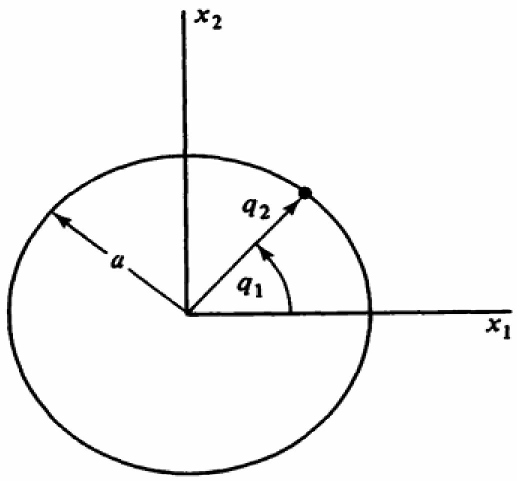


Fig. 1-1. A particle on a fixed circular path.

Let $P(x_1, x_2)$ be the cartision on-ordinat of the any point on the circle.

The constrain equation is $x_1^2 + x_2^2 = a^2$

Let a Single Generalized co-ordinate q_1 represence the one degree of freedom.

Let us define a Second Generalized co-ordinate q_2 which is constant.

$\therefore$ The transformation equation is

$$\left.\begin{aligned} x_1 &= q_2 \cos q_1 \\ x_2 &= q_2 \sin q_1 \end{aligned}\right\} \quad \dots\dots\dots\dots (1)$$

$$\frac{\partial(x_1, x_2)}{\partial(q_1, q_2)} = \begin{vmatrix} \frac{\partial x_1}{\partial q_1} & \frac{\partial x_1}{\partial q_2} \\ \frac{\partial x_2}{\partial q_1} & \frac{\partial x_2}{\partial q_2} \end{vmatrix}$$

$$= \begin{vmatrix} -q_2 \sin q_1 & \cos q_1 \\ q_2 \cos q_1 & \sin q_1 \end{vmatrix}$$

$$= -q_2[\sin^2 q_1 + \cos^2 q_1]$$

$$= -q_2(1) \neq 0$$

∴ An equation is solvable.

$$q_2 = \sqrt{x_1^2 + x_2^2}$$

∴ The Generalized co-ordinate are

$$q_1 = \sqrt{x_1^2 + x_2^2}$$

$$q_1 = \tan^{-1}(x_2/x_1)$$

4. Configuration Space:

The configuration of system of N partical is specified by giving the values of its three Cartesian co-ordinate. If the system us l independent equation constrain of the form,

$f_j(x_1; x_2 - x_{3N}, t) = \alpha_j (j = 1,2 \ldots 3N)$

Then it is possible to find n independent Generalized co-ordinates $q_1, q_2 \ldots - q_n$ where $n = 3N - l$.

Hence a set of us numbers namely the values of n q's completely a Specify the configuration of the system

This n numbers us the co-ordinate of a single point in an n-dimension Space is known as configuration Space.

1.3.Constraints:

A system of N particles may have less than 3N degrees of freedom because of the presence of constraints. These constraints put geometrical restrictions upon the possible motions of system and result in corresponding forces of constraint.

1.Constrains of restriction are condition imposed on the System:

There are classified as

(i) Holonomic constraints.

(ii) Non-Holonomic Constraints

(iii)Scleronomic constraints

(iv) Rheonomic constraints

(i) Holonomic Constraints

Suppose the configuration of a system is specified by the n generalized coordinates $q_1, q_2 \cdots, q_n$ and assume that there are k independent equations of constraint of the form

$\Phi_j(q_1, \cdots, q_n, t) = 0, j = 1,2, \cdots k$ Such is constraints called are Holonomic constraints.

A constraint which can be expressed in this system having only holonomic constraints is called holonomic System.

Example:1

Consider the motion in xy-plane of 2 partical A and B as shown in figure.

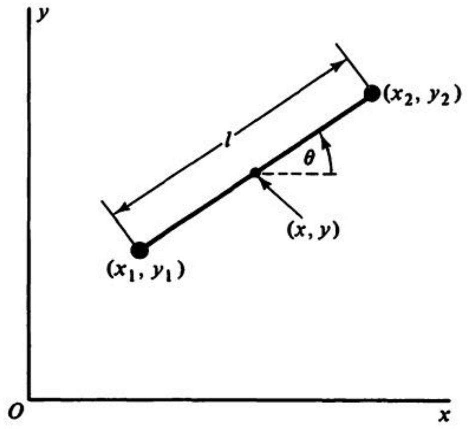


Fig. 1-2. Two particles connected by a rod of length *l*.

This particle are connected by rigid rod of length l. iss a

The equation constraints line is $(x_2 - x_1)^2 + (y_2 - y_1)^2 = l^2$.

∴ The number of degree freedom $3N - l$.

$$= 3(2) - 2$$
$$= 6 - 2 = 4.$$

(ii) Non - Holonomic constraints:

Consider the m equation of constraint express in terms of non-integrable equation of the form

$\sum_{i=1}^{n} a_{ji} d_{ji} + a_{ji} dt = 0, j = 1,2 \dots m$

Where $a_{ji}: s$ are function of q's and t.

Such constraints are called, non-holonomic constraints.

The system having only non-Holonomic constraints are called non-holonomic System.

Example:1

Consider the system of the two particle A and B connected by rigid rod AB.

Let B be mid point and θ be the angle, made by the rod with x-axis.

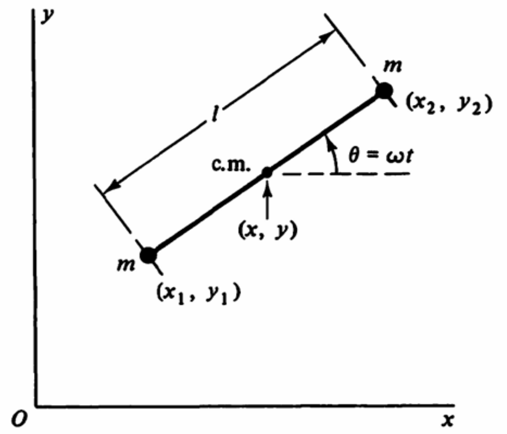


Fig 1-3 Two particles connected by a rod of length *l*

Let A and B are Supported by knife edge. Let the system move perpendicular to the direction of end in such a manner that allow the no velocity component along the rod at either particle.

The equation constraints is $\dot{x} = V\cos(90 + \theta)$

$$\Rightarrow \dot{x} = v\cos(90 + \theta)$$

$$\Rightarrow \dot{x} = v\sin\theta.$$

$$\dot{y} = v\cos\theta.$$

$$\frac{\dot{x}}{\dot{y}} = \frac{-v\sin\theta}{v\cos\theta}$$

$$\dot{x}\cos\theta = -\dot{y}\sin\theta$$

$$\therefore \dot{x}\cos\theta + \dot{y}\sin\theta = 0.$$

This expression is not exact differential equation.

∴ Hence, if is not Integrable.

∴ The equation (1) is Non-Holonomic constraints.

(iii) Scleronomic Constraints:

This constraints do not involve time t in the equation constraints as well as in the transformation equation

Example:1

A Simple pendulum in which the center Suspension is fixed

A motion of particle on a fixed wire.

The system hawing only scleronomic constraints are called scleronomic constraints System.

(iv) Rheonomic Constraints:

These constraints are express interns of equation involving time it.

Example:

(1) A simple pendulum in which center of suspension is assigned motion.

(2) A particle moving the rotating wire.

Rigid Body:

A Rigid Body is a System of particle such that a difference between any pair of particle remain constraints any time This a motion of Rigid Body is constant by the equation

$|\vec{r}_i - \vec{r}_k| =$ constant

This system is the Rheonomic System

Example:2 [Non-Holonomic constraints]

A non - Holonomic constraints when there is a rolling connect contact without steeping slipping.

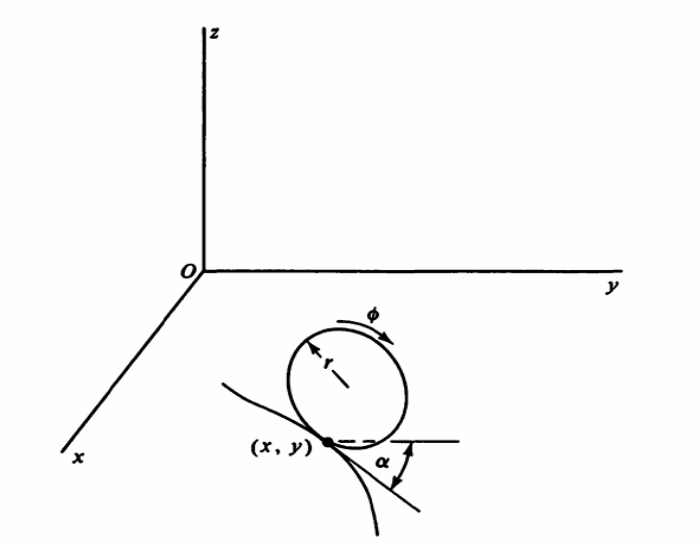


Fig 1-4 A vertical disk rolling on a horizontal plane

Let c be the centre of circlalar disc of radius r, which rolles on plane.

Let v be the velocity at any time t along the tangent at P to the curve described at the disc.

∴ The Generalized coordinate for the system where α is the angle between tangent to the part and y is the plane. ϕ is the angle of the rotation of the disc about perpendicular axis through is its center $V = r\phi'$

$$\dot{x} = r\sin\alpha$$
$$\dot{y} = v\cos\alpha.$$

$$\Rightarrow \dot{x} = r\phi' \sin\alpha$$
$$\Rightarrow \dot{y} = r\phi' \cos\alpha$$
$$\Rightarrow dx = rd\phi \sin\alpha$$
$$dy = rd\phi \cos\alpha$$
$$dx - rd\phi \sin\alpha = 0$$
$$dy - rd\phi \cos\alpha = 0.$$

This two integration are not integrable

∴ The system is not non-Holonomic

2. Define unilateral and bilateral constraints:

In the case of bilateral constraints, one imagines a small allowable displacement for any configuration of the system, the native of the displacement system is also allowable assuming any fixed value of line for such constraints they will be expressed as on equality in the case of unilateral constraint they are expressed in the form of an in equality.

Such as $f(q_1, q_2 - q_n$, t)≤ 0

This implies that configuration point is restricted to a Cartesian region. m dimensional configuration space which may vary with time.

1.4. Virtual Work:

1.Virtual displacement:

It is an infinite decimal displacement of a system and a change in the configuration of the system as the result of any arbitrary infinite decimal change of the co-ordinate δx_i.

Consistent with the form and constrain imposed on the System at the instant of time t.

During this the displacement the $3n$ cartisian co-ordinate of the system of n-partical take the variation $\delta x_1, \delta x_2 \ldots \delta x_{3N}$ without changing time They are imaginary displacement. This Small change in δx in the configuration System is known as virtual displacement.

2.Viratual Work:

Suppose the configuration of the System n partial is given in $3N$ Cartisian co-ordinate $x_1, x_2 \ldots x_{3N}$. Suppose that the force components $F_1, F_2 \ldots F_{3N}$. are applied at the corresponding co-ordinate in a positive Sign.

∴ The virtual work Sw. of this forces. In a virtual displacement, δx. is given by

$$\delta\omega = \sum_{i=1}^{3N} \vec{F_i} \cdot \delta x_i$$

(or) $\delta\omega = \sum_{i=1}^{3N} \vec{F}_i \cdot \delta\vec{r}_i$

$\vec{F_i}$ = applied force on (a_i) i^{th} particle.

$\delta\vec{r_i}$ = *position vector* of i^{th} particle

3.Define Virtual velocity:

If δx is the virtual displacement during on infinite interval of time δ_t

then $\frac{\delta x}{\delta \mathrm{t}}$ is called a virtual velocity.

Virtual work of constraints force

$\delta w_{\mathrm{c}} = \sum_{i=1}^{3N} \vec{R}_i \cdot \delta\vec{r}_i$

$\vec{R}_i$ = constraint force on i$^{\text{th}}$ particle (x_i)

$\delta\vec{r}_i$ = position vector of i$^{\text{th}}$ particle (x_i)

4.Workless constraint:

A workless constraint is any bilateral constraints such that the virtual work corresponding constraint Force is zero.

For any virtual displacement which is consistent with the constraint it can be seemed

For constant with the constraint at for system having workless constraints

That is virtual work of constraint Force is zero. ($\delta w_{\mathrm{c}} = 0$)

(i.e.,) $\sum_{i=1}^{N} \overrightarrow{R_i} \delta\vec{r_i} = 0$.

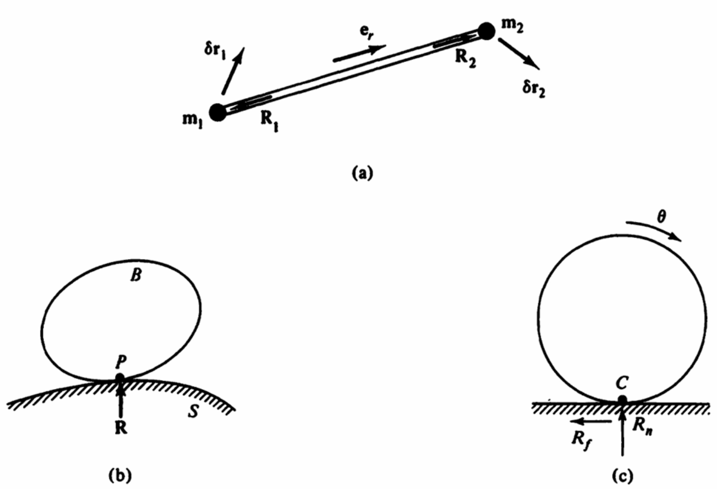


Fig 1-5 Examples of workless constraints

Example of workless constraint:

1. Rigid interconnections between particle
2. Sliding motion in a frictionless surface.
3. Rolling contact without slipping.

Theorem 1:

State and prove principle virtual work:

The necessary and sufficient condition for the static equilibrium of an initially motionless Scleronomic system which is Subject to workless constraint. (i.e.,) zero, the virtual work be done by applied force in moving through an arbitrary virtual displacement stratifies the constraint.

Proof:

Necessary part:

Let us consider the scleronomic system of N in the configuration.

If this system is in static equilibrium.

Then for each particle $\vec{F}_i + \vec{R}_i = 0$

∴ Virtual work done by all the forces in moving through an arbitrary displacement, constraint with is zero.

(i.e.,) $\delta w = \sum_{i=1}^{3N} \left(\vec{F}_i + \vec{R}_i\right) \cdot \delta\vec{r}_i = 0$ …………... (2)

If we assume that all the constraint are workless and if δr_i is reversable, Virtual displacement consistent with Constraint

$\delta_{w_c} = \sum_{i=1}^{3N} \vec{R}_i\, \delta\vec{r}_i = 0$ …………… (3)

Sub (3), in, (2).

$$\Rightarrow \delta\omega = \sum_{i=1}^{3N} \vec{F}_i\, \delta\vec{r}_i = 0$$

$$\Rightarrow \vec{F}_i = 0 \text{ for } i = 1 - 3N$$

∴ Total virtual work done by applied force is zero.

This is a Necessary condition.

Sufficient condition:

Let us assume that system of N particle is initially motionless.

But if the System not in equilibrium.

Then one is more of the particle must have net force applied to it and accordance with Newton Law of force.

Then it will start move in the direction of force.

since any motion must be compactable with the constraint.

∴ we can always the choose a virtual displacement in the direction of actual displacement actual motion at each point.

In this case the virtual work is positive

(i.e.,) $\delta\omega = \sum_{i=1}^{3N} \vec{F}_i \delta\vec{r}_i + \sum_{i=1}^{3N} \vec{R}_i \cdot \delta\vec{r}_i \geq 0.$ ………….. (4)

again the constraints are motionless then $\delta\omega_C = \sum_{i=1}^{3N} \vec{R}_i \cdot \delta\vec{r}_i = 0.$ ……….. (5)

Sub in (5) in (4)

we get $\sum_{i=1}^{3N} \vec{F}_i\, \delta\vec{r}_i > 0$ ………..(6)

Now the reversible of will give a negative work for the system.

But in any event if the system is not in equilibrium.

Then it will always possible to find a set of virtual displace constant with the constraint which will result in the virtual work of the applied force being non-zero.

∴ The System must be in equilibrium. This condition is sufficient.

Example 1:

Application of principle of virtual work.

Two frictionless blocks of equal mass M are connected by a massless rigid rod as in figure.

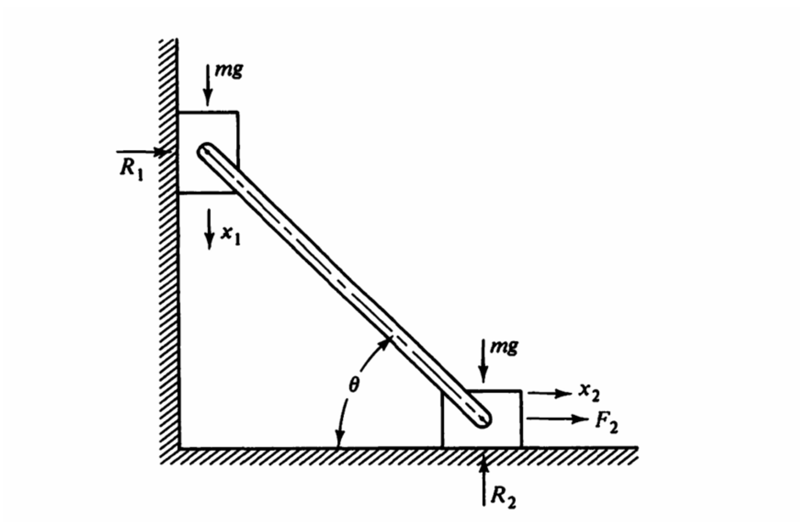


Fig 1-6 A frictionless system which is constrained to move in the vertical plane

Using x_1 and x_2 as co-ordinate solve force F_2, if the system is in static equilibrium.

Solution:

Since given system Scleronomic system with workless constraint.

The external constraint forces are the wall and flow reactions R_1 and R_2

The external constraint force is equal and opposite comprise in the rod.

∴ Total virtual work of constraint for is zero.

The applied forces are the gravitational Force acting on the blocks and the external force F_2.

∴ By principle of virtual work for static equilibrium.

$mg \cdot \delta x_1 + F_2 \delta x_2 = 0$ ………… (1)

But δx_1 and δx_2 are related by N equation of constraint.

Since that displacement components along the rod must be equal at to end.

$$\sin\theta\, \delta x_1 - \cos\theta \delta x_2 = 0$$
$$\sin\theta \delta x_1 = \cos\theta \delta x_2$$
$$\delta x_1 = \cot\theta \delta x_2$$

Sub in (1)

$$\Rightarrow\ mg\cot\theta\delta x_2 + F_2\delta x_2 = 0$$
$$(mg \cdot \cot\theta + F_2)\delta x_2 = 0$$
$$\Rightarrow\ F_2 + mg\cot\theta = 0, \delta x_2 \neq 0$$
$$\Rightarrow\ F_2 = -mg\cot\theta$$

This is the required force to keep the initially motionless system under static equilibrium.

Theorem 2:

State D' Alembert's principle:

Deduce the Lagrangian form of Alembert's principle:

Let us consider a system of N-particle. Let. $\vec{r}_i$ be the position vector of ith particle of mass m_i.

Then the equation of Motion is,

$m_i\vec{r_i} = \overrightarrow{R_i} + \overrightarrow{F_i}$ (or) $\overrightarrow{R_i} + \overrightarrow{F_i} - m_i\ddot{\vec{r_i}} = 0$

$\vec{R}_i = \cos t$ rained force

$\vec{F}_i =$ Applied force on i^{th} particle.

here $-m_i\ddot{r}_i$ is the Dimensions of the force a its known as initial force acting on the i^{th} particle.

also $\overrightarrow{R_i}$ and $\vec{F}_i$ are real or actual forces in contrast to the initial Frame.

Hence the equation is satisfied. Hence the same all forces and inertial forces acting on each particle of the system is zero.

Statement:

Force together with the other forces keep the system is in equilibrium

This is known as D'Alembert's principle

part: 2

By the principle of virtual work

the total work done by the all the forces in an arbitrary virtual displacement is zero.

$\delta_w = \sum_{i=1}^{N} \left(\vec{F}_i + \vec{R}_i - m_i\ddot{\vec{r}}_i\right) \cdot \delta\vec{r}_i = 0$ ………….. (2)

Let us assume that $\vec{R}_i$ are workers Constrained forces.

If $\delta\vec{r}_i$ are reversable displacement constro consistent with constrained.

Then $\delta\omega_c = \sum_{i=1}^{N} \vec{R}_i \cdot \delta\vec{r}_i = 0.$ .... (3)

Sub (3) in (2).

$$\delta\omega = \sum_{i=1}^{N} [\vec{F}_i - m_i\ddot{\vec{r}}_i] \cdot \delta\vec{r}_i = 0.$$

This is known as Lagrange form of D ' Alembert's principle.

Example 2:

Application of D' Alembert's principle:

A particle of mass m is suspended by a massless wire of length $r = a + b\cos\omega t$, $a, b > 0$. To form a spherical pendulum, find the equation of motion.

Proof:

Let us use spherical polar coordinate

$\mathrm{x} = a\cos\theta\, sin\,\phi \quad , \qquad 0 \le \gamma \le \infty$

$y = a\sin\theta\, sin\,\phi \quad , \quad 0 \le \theta \le 2\pi$

$z = a\cos\phi \;, \quad 0 \le \phi \le \pi$

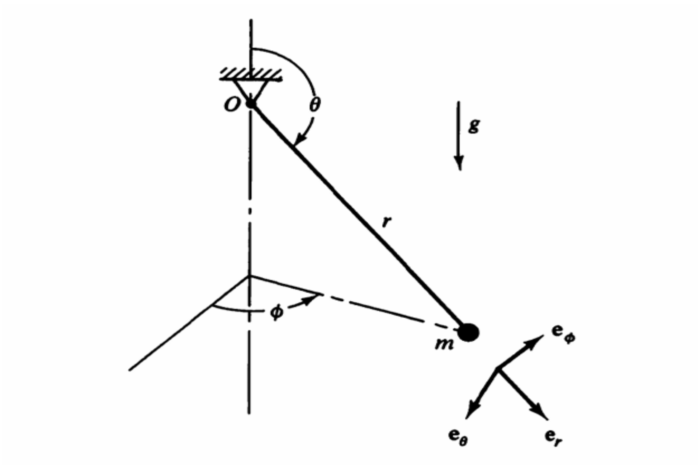


Fig. 1-7. A spherical pendulum of variable length.

Let $P(r, \theta, \phi)$ be the position of mass m at time then the acceleration of P is given by,

$$\left.\begin{aligned} \ddot{\vec{r}} = \; & \left(\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta\right)\vec{e_r} \\ & +\left(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta\right)\vec{e}_\theta \\ & +(r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta)\vec{e_\phi} \end{aligned}\right\} \ldots\ldots\ldots\ldots(1)$$

Where $\vec{e_r}, \vec{e_\theta}, \vec{e_\phi}$ are the unit vector forming the an orthogonal .

A Virtual displacement with the Constrained

$\Rightarrow \delta\vec{r} = r \cdot \overrightarrow{\delta\theta}\overrightarrow{e_\theta} + r\sin\theta\delta\phi\overrightarrow{e_\phi}$ .... (2)

The applied gravitational force is $\vec{F} = -mg\cos\theta\vec{e}_r + mg\sin\theta\vec{e}_\theta$ ....(3)

By D' Alembert's principle.

$$\sum_{i=1}^{N}(\vec{F}_i - M_i\ddot{\vec{r}}_i) \cdot \delta\vec{r}_i = 0$$

$$\Rightarrow \sum_{i=1}^{N}(\vec{F}_i \cdot \delta\vec{r}_i - m_i\ddot{\vec{r}}_i \cdot \delta\vec{r}_i) = 0$$

$$\because \vec{F}_i\delta\vec{r}_i - m_i\ddot{\vec{r}}_i \cdot \delta\vec{r}_i = 0 \quad \ldots\ldots\ldots (4)$$

Sub (1)(2)(3) in (4)

$$(mgr\sin\theta - mr^2\dot{\theta}' - m^2r\dot{r}\dot{\theta} + mr^2\dot{\phi}^2\sin\theta\cos\theta)\delta\theta$$
$$+\left(-mr^2\dot{\phi}'\sin^2\theta - m^2r\dot{\phi}\sin^2\theta - m^2r^2\dot{\theta}\dot{\phi}\sin\theta\cos\theta\right)\delta\phi = c$$
$$\Rightarrow mr(g\sin\theta - r\dot{\theta}' - 2\dot{r}\dot{\theta} + r\dot{\phi}^2\sin\theta\cos\theta)\delta\theta$$
$$-mr\sin\theta(r\dot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta)\delta\phi = 0.$$

But $\delta\theta$ and $\delta\phi$ are independent.

Equate the co-eff $\delta\theta$ and $\delta\phi$ is equal zero

we get, $g\sin\theta - r\ddot{\theta} - 2r\dot{\theta} + r\dot{\phi}^2\sin\theta\cos\theta = 0$

and $(r\dot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta) = 0.$ .... (5)

But given

$$\gamma = a + b\cos\omega t$$
$$\dot{\gamma} = -b\omega\sin\omega t$$
$$\ddot{\gamma} = -b\omega^2\cos\omega t.$$

we get, sub in (5),

$$(a + b\cos\omega t)\ddot{\theta} - 2b\omega\sin\omega t\dot{\theta} - (a + b\cos\omega t)\dot{\phi}^2\sin\theta\cos\theta = g\sin\theta$$

$$(a + b\cos\omega t)\ddot{\phi}\sin\theta - 2b\omega\sin\omega t\dot{\phi}\sin\theta + 2(a + b\cos\omega t)\dot{\theta}\dot{\phi}\cos\theta = 0.$$

This are the required equation of motion.

5.Define Generalized Force:

Let us consider system of N-particle with $3N$ Cartesian co-ordinate $(x_1, x_2 \cdots - x_{3N})$. Let $(F_1, F_2, F_3 \ldots F_{3N})$ are the applied force Corresponding to the co-ordinate $(x_1, x_2 \ldots\ldots x_{3N})$ in the positive sense.

The virtual work δw of the this force in a virtual displacement δx is given by,

$\delta w = \sum_{i=1}^{3} \vec{F}_i \, \delta\vec{x_i}$ ……………. (1)

Now the Cartesian co-ordinate $(x_1, x_2 \dots x_{3N})$ are related n-generalised co-ordinate $(q_1, q_2 \dots q_n, t)$

(i.e.,), The transformation is given by,

$x_k = x_k(q_1, q_2 \dots q_n, t)\; k = 1, \cdots 3N$

∴ The total differential coefficient of x_k is

$$d_{x_k} = \frac{\partial x_k}{\partial q_1} dq_1 + \frac{\partial x_k}{\partial q_2} dq_2 + \cdots + \frac{\partial x_k}{\partial q_n} dq_n + \frac{\partial x_k}{\partial t} dt$$

$$dx_k = \sum_{j=1}^{n} \frac{\partial x_k}{\partial q_i} \partial q_i + \frac{\partial x_k}{\partial t} dt,\; k = 1,2 \dots 3N.$$

It is converted into virtual displacement, $\delta x_k = \sum_{j=1}^{n} \frac{\partial x_k}{\partial q_j} \delta q_j$……..(2) for $k = 1,2 \dots 3N$

A δq are displaced by dq_i and omitted the dt term.

∵ the time is held fixed. During the virtual displacement.

Sub (2) in (1)

$$\delta\omega = \sum_{i=1}^{3N} \sum_{j=1}^{n} \vec{F}_i \frac{\partial x_i}{\partial q_j} \delta q_j$$

(or) $= \sum_{j=1}^{n} Q_j \delta q_j$ where $Q_i = \sum_{i=1}^{3N} F_i \frac{\partial x_i}{\partial q_j}$ for $i = 1,2 \dots 3$N

is called generalized forces.

Example 3:

Application of Generalized force:

The three particle are connected by two rigid rod having a fixed point between them to form a system of in Figure. A vertical force F and a moment m are applied as shown. If

$$x_1 = q_1 + q_2 + \frac{q_3}{2}$$
$$x_2 = q_1 - q_3$$
$$x_3 = q_1 - q_2 + \frac{1}{2} q_3$$

Find the Generalized co-ordinate and Generalized Force.

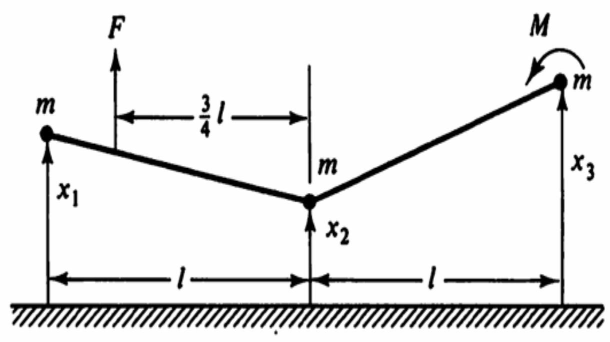


Fig. 1-8. A system with an applied force and moment.

Proof:

Given transformation equation are,

$x_1 = q_1 + q_2 + \frac{q_3}{2}$

$x_2 = q_1 - q_3$

$x_3 = q_1 - q_2 + \frac{1}{2} q_3$

To find, $J = \left(\frac{x_1, x_2, x_3}{q_1, q_2, q_3}\right)$

$$\frac{\partial(x_1, x_2, x_3)}{\partial(q_1 q_2, q_3)} = \begin{pmatrix} \frac{\partial x_1}{\partial q_1} & \frac{\partial x_2}{\partial q_1} & \frac{\partial x_3}{\partial q_1} \\ \frac{\partial x_1}{\partial q_2} & \frac{\partial x_2}{\partial q_2} & \frac{\partial x_3}{\partial q_3} \\ \frac{\partial x_1}{\partial q_3} & \frac{\partial x_2}{\partial q_2} & \frac{\partial x_3}{\partial q_3} \end{pmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1/2 & -1 & -1/2 \end{vmatrix} = 1(0-1) - 1\left(\frac{1}{2} + \frac{1}{2}\right) + 1(-1-0) = -1 - 1 - 1$$

$$= -3 \neq 0$$

∴ Transformation exists.

Solve (1), (2), (3)

$(1) + (2) + (3) \Rightarrow 3q_1 = x_1 + x_2 + x_3$

$$\Rightarrow q_1 = \frac{1}{3}(x_1 + x_2 + x_3)$$

$(1) - (3) \Rightarrow 2q_2 = x_1 - x_3$

$$q_2 = \frac{1}{2}(x_1 - x_3)$$

Sub in equation (2)

$$x_2 = \frac{1}{3}(x_1 + x_2 + x_3) - q_3$$
$$q_3 = \frac{1}{3}[x_1 + x_2 + x_3] - x_2$$
$$q_3 = \frac{1}{3}[x_1 - 2x_2 + x_3]$$

(Given figure)

To Find generalized force,

Let the force F can be replaced by a force $\frac{3}{4}$ at x_1 and $F/4$ at x_2

The moment M can be replaced equlant opposite force of magnitude $\frac{m}{l}$ acting at the direction of x_2 & x_3.

$$F_1 = \frac{3F}{4}$$
$$F_2 = \frac{F}{4} - \frac{M}{l}$$
$$F_3 = \frac{M}{l}$$

∴ Let Q_1, Q_2, Q_3 be the Generalised Force.

$$Q_1 = F_1 \frac{\partial x_1}{\partial q_1} + F_2 \frac{\partial x_2}{\partial q_1} + F_3 \frac{\partial x_3}{\partial q_1}$$
$$= \frac{3F}{4}(1) + \left(\frac{F}{4} - \frac{M}{l}\right)(1) + \frac{M}{l}(1)$$
$$= \frac{3F}{4} + \frac{F}{4} - \frac{M}{l} + \frac{M}{l}$$
$$= \frac{4F}{4}$$
$$Q_1 = F$$

$$Q_2 = F_1 \frac{\partial x_1}{\partial q_2} + F_2 \frac{\partial x_2}{\partial q_2} + F_3$$
$$= \frac{3F}{4}(1) + \left(\frac{F}{4} - \frac{M}{l}\right)(0) + \frac{M}{l}(-1)$$
$$= \frac{3F}{4} + 0 - \frac{M}{l}$$
$$Q_2 = \frac{3F}{4} - \frac{M}{l}$$

$$Q_3 = F_1 \frac{\partial x_1}{\partial q_3} + F_2 \frac{\partial x_2}{\partial q_3} + F_3 \frac{\partial x_3}{\partial q_3}$$
$$= \frac{3F}{4}\left(\frac{1}{2}\right) + \left(\frac{F}{4} - \frac{M}{l}\right)(-1) + \frac{M}{l}\left(\frac{1}{2}\right)$$

$$= \left(\frac{3F}{8} - \frac{F}{4}\right) + \left(\frac{M}{l} + \frac{M}{2l}\right)$$
$$= \left(\frac{3F - 2F}{8}\right) + \left(\frac{2M + M}{2l}\right)$$
$$Q_3 = \frac{F}{8} + \frac{3M}{2l}$$

∴ The generalized force is,

$$F, \frac{3F}{4} - \frac{M}{l}, \frac{F}{8} + \frac{3M}{2l}.$$

1.5. Energy and Moment:

Find Energy and Moment Conservation Force:

Let $P(x, y, z)$ be the position of single particle in space.

Let F be total force acting on the particle has on components.

$$F_x = -\frac{\partial v}{\partial x}$$
$$F_y = -\frac{\partial v}{\partial y}$$
$$F_z = -\frac{\partial v}{\partial z}$$

where $V(x, y, z)$ is the potential energy (i.e.,) It's not a function of velocity and of time

A force F meeting this condition is known as conservative force.

The condition for conservative Force:

Consider work done by the force $\vec{F}$ as it moves through an infinite decimal displacement $d\vec{r}$.

$$d\omega = \vec{F} \cdot \vec{d}$$
$$= \left(F_x\vec{\imath} + F_y\vec{\jmath} + F_3\vec{k}\right) \cdot (\vec{\imath}dx + \vec{\jmath}dy + \vec{k}dz)$$
$$= F_x dx + F_y dy + F_z dz$$
$$= \frac{-x}{\partial x} dx + \frac{\partial \pi}{\partial y} dy + \frac{\partial \pi}{\partial z} dz$$
$$d\omega = dF - dv$$

$d\omega$ is an exact differential equation.

Now consider work done w by a force $\vec{F}$ as an particle moves over a certain path between the points A and B.

$$w = \int_A^B \vec{F} \cdot \overrightarrow{dr}$$
$$= \int^B - dv = -[v]_A^B$$
$$= -[V_B - V_A]$$
$$W = [V_A - V_B]$$

Since P.E V is a function of position only. We conclude that work done on the particle depends upon initial and finial position.

If A & B are co-inside, then the work done in moving around any closed path zero.

(i.e.,) $\oint_C \vec{F} \cdot d\vec{r} = 0,$

Where $\vec{F}$ is conservative Force.

This is a condition for the. force to be conservative.

Theorem 1:

State and prove principle of work and Kinetic Energy

The increasing $K.E$ of a particle as in moves from one arbitrary point two another is equal to the work done by the force acting on the particle during the given inter

Proof:

Let us define kinetic energy T of a particle of mass m is given by, $T = \frac{1}{2}mv^2$:

Where v is the velocity of particle relative to the inertial frame.

consider the integral,

$$w = \int_A^B \vec{F} \cdot d\vec{r}$$

Which gives the of work done on the particle By the force $\vec{F}$ as the partide moves from A to B.

By Newton law of motion:

$$\vec{F} = m\ddot{\vec{r}}$$
$$\omega = \int_A^B m\ddot{\vec{r}} d\vec{r}$$
$$= \int_A^B m\ddot{\vec{r}} \cdot \frac{dr}{dt} dt$$
$$= \frac{m}{2} \int_A^B d(\dot{\vec{r}} \cdot \dot{\vec{\gamma}})$$
$$= \frac{m}{2} \int_A^B d(v^2).$$

$$= \frac{m}{2}[v^2]_A^B$$
$$= \frac{m}{2}[v_B^2 - v_A^2]$$
$$= \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2$$
$$w = T_B - T_A$$

This will be true whether force is Conservative (a ri not.

State the principle of conservation Energy

If the only force acting on the given particle are conservative.

Then $V_A - V_B = T_B - T_A$

$\Rightarrow T_A + V_A = T_B + V_B$

(i.e.,) The points A & B are arbitrary. The principle of conservation of energy state that "The total mechanical Energy E remains constant during the motion of the particle.

Theorem 2: (Equilibrium configuration)

An equilibrium configuration of a conservative holonomic system with workless fixed constraint must occur at a position where the P.E have a stationary value.

Proof:

Let $F_1, F_2 \dots F_n$ are the applied force. acting on n-particle. They are conservative.

Let $V = V(x_1 \dots x_n)$ be the P.E

∴ The virtual work of the applied Force is $\delta\omega = \sum_{j=1}^{3N} F_j \delta x_j$

$$= -\sum_{j=1}^{3N} \frac{\partial v}{\partial x_j}, x_j$$
$$\delta w = -\delta v$$

∴ By using principle of virtual work,

$$\delta w = 0$$
$$\Rightarrow \delta v = 0.$$

⇒ is the condition of static equilibrium.

For every virtual displacement Consistent with constraint.

The P.E is expressed in terms generalized coordinate $q_1, q_2 \dots\dots q_n$.

$$\delta v = \sum_{j=1}^{n} \frac{\delta v_i}{\partial q_i} \delta q_j$$

The condition that $\delta v = 0$

For an arbitrary virtual displacement requires that the co-efficient must be zero. At the equilibrium configuration

$$\Rightarrow \sum_{i=1}^{n} \frac{\partial v}{\partial q_i} = 0$$

$$\Rightarrow \frac{\partial v}{\partial q_i} = 0 \text{ for } i = 1,2 \ldots n.$$

T $\Rightarrow P.E$ is stationery value.

Theorem 3:

State and prove Konig's Theorem (or) Resolving Theorem of Kinetic Energy

The total Kinetic Energy of the system is equal to. Sum of

(i) The Kinetic Energy due to a particle having a mass equal to the total mass of the system. and Moving with the velocity of the center mass.

(ii) The Kinetic Energy due to the motion of System relative to its center of mass

Proof:

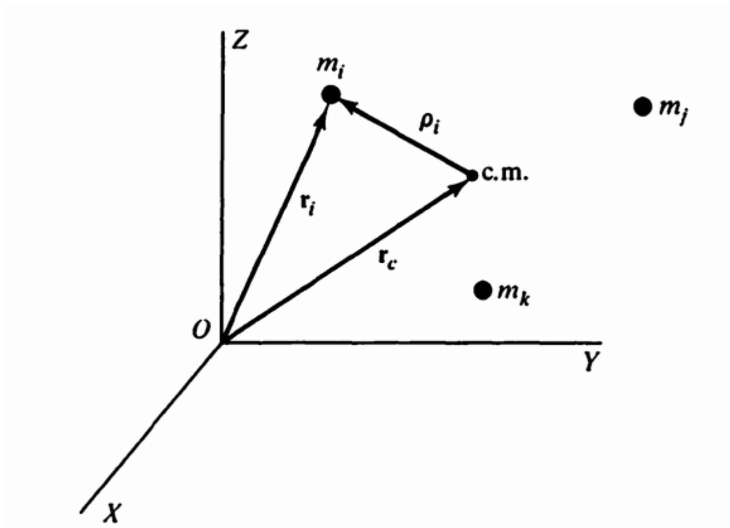


Fig 1-9 Position vectors for a system of particles

Let us consider a system of N-partide Let $\vec{r}_i$ be the position vector of i^{th} particle of mass m_i relative to initial frame. Let $\vec{r}_c$ be the position vector of center of mass cm, with respect to 0 . Let ρ_i be the position vector. m_i relative to cm. Total Kinetic Energy of the System is equal to

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i \dot{\vec{r}}_i^2$$

Sum of individual Kinetic Energy of particle.

From Figure $\vec{r}_i = \vec{r_c} + \vec{p_i}$

$$\dot{\vec{r_i}} = \dot{\vec{r_c}} + \dot{\vec{p_i}}$$

$$\therefore T = \frac{1}{2}\sum_{i=1}^{N} m_i\left(\dot{\vec{r_c}} + \dot{\vec{p_i}}\right)^2$$

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i\dot{\vec{r_c}}$$

Since ρ_i is measure from center of mass cm also $\dot{\vec{r_c}}$ does not entre with the summation and can be factor out.

$$\text{(ie)} \sum_{i=1}^{N} m_i\dot{\vec{p}}_i = 0 \text{ for } \dot{\vec{r_c}} \neq 0$$

$$\therefore T = \frac{1}{2}m\dot{\vec{r}}_c^2 + \frac{1}{2}\sum_{i=1}^{N} m_i\dot{\vec{p}}_i^2$$

Theorem 4: (Kinetic Energy of a Rigid Body)

The with usual notation that the rotational kinetic energy can be written in the form of,

$$T_{\text{rot}} = \frac{1}{2}\overline{\omega^T} I\,\vec{\omega}$$

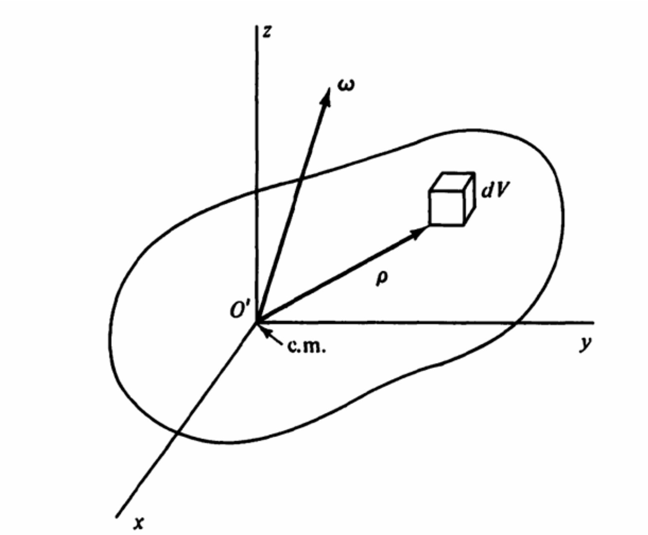


Fig 1-10 A typical volume element in a rotating rigid body

Let o be the center of mass (c. m)

Let dv be the small volume element

Let $\vec{\rho}$ be the of small volume element dv relative to 0 .

Let $\vec{\omega}$ be the angular velocity of the rigid body about an axis trough 0 .

The Let $\vec{\rho}$ be the density/unit mass.

∴ The mass of the small volume element is ρdV.

The dimensions of typical volume element can be chosen to be so small that its rotational kinetic energy is negligible compared with translational kinetic energy.

In the limit, each element of a rigid body can be considered as a particle of infinitesimal mass.

The first term on the right side is called the translational kinetic energy of rigid body the second term is rotational kinetic energy.

$$\therefore T = \frac{1}{2} m\dot{\vec{r}}_c{}^2 + T_{\text{rot}}$$

To Find Trot:

We know that $\dot{\vec{\rho}} = \vec{\omega} \times \vec{\rho}$ ……….(2)

$$\dot{\vec{\rho}}^2 = \dot{\vec{\rho}} \cdot \dot{\vec{\rho}}$$

$$= (\vec{\omega} \times \vec{\rho}) \cdot \dot{\vec{\rho}}$$

$$= \vec{\omega} \cdot (\vec{\rho} \times \dot{\vec{\rho}})$$

$$= (\vec{\omega} \cdot \vec{\rho}) \times \vec{\rho}$$

$$\dot{\vec{\rho}}^2 = (\vec{w} \cdot \vec{\rho}) \times (\vec{\omega} \times \vec{\rho})$$

$$\rho\dot{\vec{\rho}}^2 = \vec{\omega} \cdot [\rho\vec{\rho} \times (\vec{w} \times \vec{\rho})]$$
$$= \vec{\omega} \cdot [\rho(\rho^2\vec{\omega} - (\vec{\rho} \cdot \vec{\omega})\vec{\rho}]$$

$$\therefore T_{\text{rot}} = \frac{1}{2}\int_v \rho\dot{\vec{\rho}}^2 dv$$
$$= \frac{1}{2}\vec{\omega} \cdot \int_v \rho[\rho^2\vec{\omega} - (\vec{\rho}, \vec{w})\vec{\rho}]dv \quad \dots\dots\dots\dots (3)$$

Let $\vec{\rho} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$

$\vec{w} = w_x\vec{\imath} + w_y\vec{\jmath} + w_z\vec{k}$

To find: $\rho^2\vec{\omega} - (\vec{\rho} \cdot \vec{\omega})\vec{\rho}$.

$$= (x^2 + y^2 + z^2)(w_x\vec{\imath} + w_y\vec{\jmath} + w_z\vec{k}) - (xw_x + yw_y + zw_z)(x\vec{\imath} + y\vec{\jmath} + z\vec{k})$$

$$= x^2\omega_x\vec{\imath} + y^2w_x\vec{\imath} + z^2w_x(\vec{\imath}) + x^2w_y\vec{\jmath} + y^2w_y\vec{\jmath} + z^2wy\vec{\jmath} + x^2w_z\vec{k} + y^2w_z\vec{k} + z^2w_z\vec{k}$$

$$-x^2\omega_x\vec{\imath} - xy\omega_y\vec{\imath} - xz\omega_z\vec{\imath} - xy\omega_x\vec{\jmath} - y^2\omega_y\vec{\jmath} - yz\omega_z\vec{\jmath} - zx\omega_x\vec{k} - yz\omega_y\vec{c} - z^2\omega_z\vec{k}$$

$$= \{(y^2 + z^2)\omega_x - xy\omega_y - xz\omega_z\}\vec{\imath} + \{(z^2 + x^2)w_y - yzw_z - yx_5w_xx\}\vec{\jmath}$$
$$+ \{(x^2 + \dot{y}^2)\omega_z - zx\omega_x - zy\omega_y\}\vec{k}.$$

Sub in equation (3)

$$T_{\text{rot}} = \frac{1}{2}\left(\omega_x\vec{i} + w_y\vec{j} + w_z\vec{k}\right) \cdot \int_V \rho[(y^2 + z^2)w_{\text{x}} - x_y w_y - xzw_z)\vec{i}]$$

$$+\{\{(z^2 + x^2)w_y - yz\omega_z - yx\omega_x\big)\vec{j}\}$$

$$+[\{x^2 + y^2)\omega_z - zx\omega_x - zy\omega y\}\vec{k}]dv$$

$$T_{\text{rot}} = \frac{\rho}{2}\int_V [(y^2 + z^2)w_{\text{x}}{}^2 - x_y w_{\text{x}} w_y - xzw_{\text{x}} w_z)\vec{i}]$$

$$+\{\{(z^2 + x^2)w_{\text{y}}{}^2 - yzw_y\omega_z - yx\omega_x w_y\big)\vec{j}\}$$

$$+\big[\{x^2 + y^2)w_{\text{z}}{}^2 - zx\omega_x\omega_z - zyw_y\omega_z\}\vec{k}\big]dv$$

$$T_{\text{rot}} = \frac{1}{2}I_{xx}\omega_x{}^2 + \frac{1}{2}I_{yy}\omega_y^2 + \frac{1}{2}I_{zz}w_z^2 + I_{xy}w_x w_y + I_{yz}w_y w_z + I_{zx}w_z w_x$$

$$= \frac{1}{2}\sum_i \sum_j I_{ij}\omega_i\omega_j$$

$$T_{\text{rot}} = \frac{1}{2}\vec{\omega}^{\mathsf{T}} I\vec{\omega}$$

Where,

$$I_{xy} = I_{yx} = -\int_v \rho xydv$$

$$I_{yz} = I_{zpy} = -\int_V \rho yzdv$$

$$I_{zx} = I_{xz} = -\int_v \rho xzdv$$

$$I_{xx} = \int_v \rho(x^2 + z^2)dv$$

$$I_{yy} = \int_v \rho(x^2 + z^2)dv$$

$$I_{zz} = \int_v \rho(x^2 + y^2)dv$$

$$\vec{\omega} = \begin{pmatrix} w_x \\ w_y \\ w_z \end{pmatrix}, I = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

Note:

Suppose that axis of the rotation through the center of mass is chosen along any the coordinate axis. $\therefore\ T_{\text{rot}} = \frac{1}{2}I\vec{\omega}^2$. where I is the moment of inerisia about an axis which is in direction of $\vec{\omega}$. Through centre of mass.

Example 1:

Find the kinetic energy system of particle in terms of motion with respect to an arbitrary fixed point.

Proof:

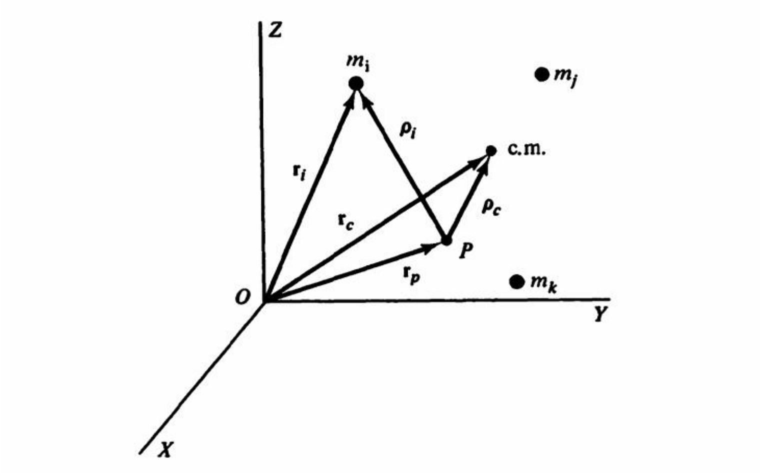


Fig 1-11 Position vector for a system of particles, using an arbitrary reference point

$\vec{r}_P$ be the position vector of P w. $r.$ to O

$\vec{r}_C$ be the position vector of c.m $w \cdot r.$t O

$\vec{r}_i$ be the position vector of m_i w.r. t. O

$\vec{\rho}_c$ be the position vector of center of mass w. $r.$ to ρ.

$\vec{\rho}_i$ be the position vector of m_i w.r.t. ρ

From Figure:

$$\vec{r}_i = \vec{r}_p + \vec{p}_i$$
$$\dot{\vec{r}}_i = \dot{\vec{r}}_p + \dot{p}_i$$

Kinetic Energy $T = \frac{1}{2}\sum_{i=1}^{N} m_i \dot{\vec{r}}_i^2$

$$= \frac{1}{2}\sum_{i=1}^{N} m_i(\dot{\vec{r}}_p + \vec{P}_i)^2$$

$$= \frac{1}{2}\sum_{i=1}^{N} m_i\left[\dot{\vec{r}}_p^2 + \dot{\vec{P}}_i^2 + 2\dot{\vec{r}}_p\dot{\vec{p}}_i\right]$$

$$= \frac{1}{2}\sum_{i=1}^{N} m_i\dot{\vec{r}}_p^2 + \frac{1}{2}\sum_{i=1}^{N} m_i\dot{p}_i^{\,2} + \sum_{i=1}^{N} m_i\vec{r}_p\dot{\vec{p}}_i \qquad \ldots\ldots\ldots\ldots (1)$$

Since center mass rotate about any point P.

$$\vec{P}_c = \frac{\sum_{i=1}^{N} m_i\vec{P}_i}{\sum_{i=1}^{1} m_i}$$

$m\overrightarrow{\rho_c} = \sum_{i=1}^{N} m_i \vec{P}_i$

$$m \cdot \dot{\vec{p}}_c = \sum_{i=1}^{N} m_i \dot{\vec{p}}_i$$

Sub in (1)

$$\Rightarrow T = \frac{1}{2} m \dot{\vec{r}}_p^2 + \frac{1}{2} \sum_{i=1}^{N} m_i \dot{\vec{p}}_i^2 + m \dot{\vec{r}_p} (m \dot{\vec{p}}_c)$$

$T = T_1 + T_2 + T_3$

$T_1 = \frac{1}{2} m \dot{\vec{r}}_p^2$

where, $T_2 = \frac{1}{2} \sum_{i=1}^{N} m_i \dot{\vec{P}}_i{}^2$

$T_3 = \dot{\vec{r}_p} \sum_{i=1}^{N} m_i \dot{\vec{P}_i}$

T_1: kinetic energy due a particle having mass M w.r.t P.

T_2 : kinetic energy of the system due to its motion w.r.t P.

T_3 = The secalar product of the velocity of the reference point and the linear momentum of the system relative reference point.

Definition: Angular momentum

Consider a particle of mass m. moving with velocity V. Then the angular momentum of m. is defined as $\vec{H} = \vec{r} \times m\vec{r}$

Theorem: 1

State Resolution for angular momentum of system of particle.

The angular momentum of system of particles of total mass m about a fixed point O is equal to the angular momentum of single particle of mass m moving with center of mass plus the angular momentum of system about center of mass.

Proof:

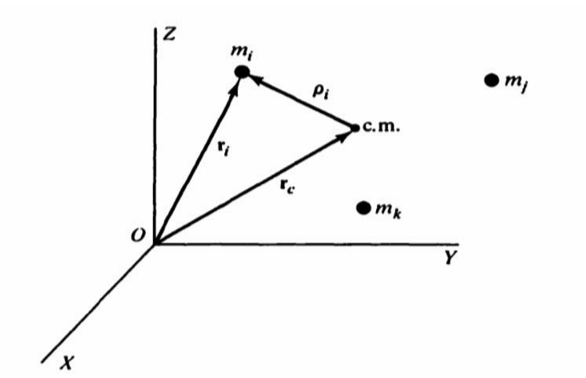


Fig 1-12

Let us consider system of N-particle. (as in figure.)

Let $\vec{r_i}$ be the position vector of m_i w.r.t O

Let $\vec{r_c}$ be the position vector of center of mass w.r.t O

Let $\vec{\rho_i}$ be the position vector of m_i w.r.t center of mass

from figure, $\vec{r_i} = \vec{r_c} + \vec{\rho_i}$

$$\vec{r_i} = \dot{\vec{r_c}} + \dot{\vec{\rho_i}}$$

By the definition total angular momentum $\vec{H}$ of the system w.r.t O is = sum of momentum of individual linear momentum of the particle w.r.t O.

$$\vec{H} = \sum_{i=1}^{N} \vec{r_i} \times m_i \vec{r_i}$$

$$= \sum_{i=1}^{N} (\vec{r_c} + \vec{p_i}) \times m_i(\dot{\vec{r_c}} + \dot{\vec{p_i}})$$

$$= \left(\vec{r_c} + \sum_{i=1}^{N} \vec{P_i}\right) \times \left(\dot{\vec{r_c}} \sum_{i=1}^{N} m_i + \sum_{i=1}^{N} m_i \dot{\vec{P_i}}\right)$$

$$= (\vec{r_c} \times \dot{\vec{r_c}} \cdot m) + \vec{r_c} \times \sum_{i=1}^{N} m_i \dot{\vec{\rho_i}} + \sum_{i=1}^{N} \vec{\rho_i} \times \dot{r_c} \sum_{i=1}^{N} m_i + \sum_{i=1}^{N} \vec{p_i} \times \sum_{i=1}^{N} m_i \dot{\vec{p_i}}$$

$$\vec{H} = (\vec{r_c} \times \dot{\vec{r}}_c m) + \dot{\vec{r_c}} \times \sum_{i=1}^{N} m_i \dot{\vec{p_i}} + \sum_{i=1}^{N} m \vec{p_i} \times \dot{\vec{r_c}} + \sum_{i=1}^{N} \left(\vec{p_i} \times \dot{\vec{P}}_i m\right)$$

Since ρ_i is measure from com.

$$\vec{P_c} = \frac{\sum_{i=1}^{n} m_i \vec{\rho_i}}{\sum_{i=1}^{m} m_i}$$

$$m\vec{p_c} = \sum_{i=1}^{n} m_i \vec{\rho_i}$$

$$m\dot{P_c} = \sum_{i=1}^{N} m_i \dot{\vec{\rho_i}}$$

$$0 = \sum_{i=1}^{N} m_i \dot{\vec{\rho_i}} \qquad [\because \rho_c = 0] \quad [here\ \rho_c = \text{ constant } = 0]$$

$$\vec{H} = (\vec{r_c} \times m\dot{\vec{r_c}}) + \sum_{i=1}^{N} \vec{\rho_i} \times m_i \dot{\vec{\rho_i}}$$

Hence angular momentum of system about O.

= angular momentum of center of mass about O

+ angular momentum of the system w.r.t center of mass

Example 1:

Find the angular momentum of a rigid body about O.

Proof:

We know that

The Angular momentum of the System about O.

is $\vec{H} = (\vec{r_c} \times m\dot{\vec{r_c}}) + \sum_{i=1}^{N} \vec{\rho_i} \times m_i\dot{\vec{\rho_i}}$

Where the second term is angular momentum of the system w.r.t center of mass.

$$\vec{H}_c = \sum_{i=1}^{N} \vec{P_i} \times m_i\dot{\vec{\rho}}_i$$

(i.e.,) $\quad = \int_v \vec{\rho} \times \rho dv \dot{\vec{\rho}}$

$$H_c = \int_v \rho[\vec{\rho} \times (\vec{\omega} \times \vec{\rho})]dv.$$

∴ kinetic energy of the rigid, Body.

$$T_{\text{rot}} = \frac{1}{2}\vec{\omega}\int_v \rho[\vec{\rho} \times (\vec{\omega} \times \vec{\rho})]dv$$

$$T_{\text{rot}} = \frac{1}{2}\vec{\omega}\vec{H}_c$$

Example 2:

Find the angular momentum of a rigid body with respect to arbitrary point P

Proof:

We know that, the angular momentum of the rigid body about is $\vec{H} = \sum_{i=1}^{N} \vec{r}_i \times m_i\dot{\vec{r_i}}$

We know that the angular momentum of the rigid body w.r.t arbitrary point P.

$$\vec{H}_p = \sum_{i=1}^{N} \vec{\rho}_i \times m_i\dot{\vec{\rho_i}}$$

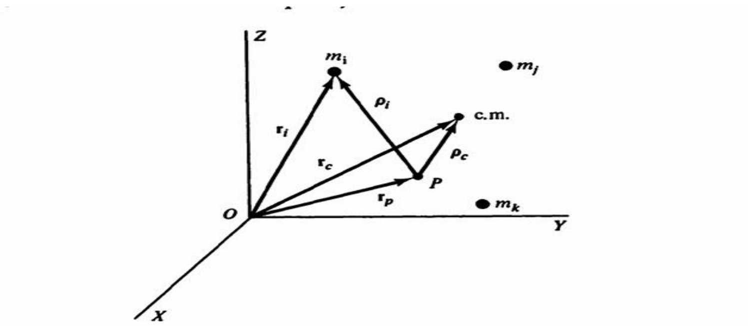


Fig 1-13

from figure:

$$\vec{r_i} = \vec{r_p} + \vec{\rho}_i$$
$$\vec{r_c} = \vec{r_p} + \vec{p_c}$$
$$\vec{\gamma_i} - \vec{r_c} = \vec{p_i} - \vec{p_c}$$
$$\Rightarrow \vec{p_i} = \vec{r_i} - \vec{r_c} + \vec{\rho_c}$$
$$\Rightarrow \dot{\vec{\rho_i}} = \dot{\vec{r_i}} - \dot{r_c} + \dot{\vec{\rho_c}}$$
$$\vec{H_p} = \sum_{i=1}^{N} (\vec{r_i} - \vec{r_c} + \vec{\rho_c}) \times m_i(\dot{\vec{r_i}} - \dot{\vec{r_c}} + \dot{\vec{\rho_c}})$$
$$\vec{H_p} = \sum_{i=1}^{N} \vec{r_i} \times m_i\dot{\vec{r_i}} - r_c \times \sum_{i=1}^{N} m_i\dot{\vec{r_c}} + \vec{\rho_c} \times \sum_{i=1}^{N} m_i\dot{\vec{\rho}}_c$$
$$\vec{H}_p = \vec{H} - \vec{r}_c \times m\dot{\vec{r}}_c + \vec{\rho}_c \times m\dot{\vec{\rho}}_c$$

Generalized Momentum (p)

Definition:

Consider a system of n particle with generalized coordinates $q_1, q_2 \ldots\ldots q_n$.

Let T = kinetic energy = $T(q, \dot{q}, t)$

V = potential energy = $V(q, t)$.

We find Lagrangian function $L = L(q, \dot{q}, t)$

$L = T - V$

Now we define generalized momentum (p_i) associated with generalized Co-ordinate with q_i.

As $p_i = \frac{\partial L}{\partial \dot{q}_i}$

$$P_i = \frac{\partial}{\partial \dot{q}_i}(T - V)$$
$$= \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial v}{\partial \dot{q}_i}$$
$$P_i = \frac{\partial T}{\partial \dot{q}_i} - 0$$
$$P_i = \frac{\partial T}{\partial \dot{q}_i} \qquad (\because V = V(q, t)]$$

Example :1

Three particles are connected by two rigid rod having a join b/w them to form the system as shown. a vertical force F and applied momentum M as in Figure.

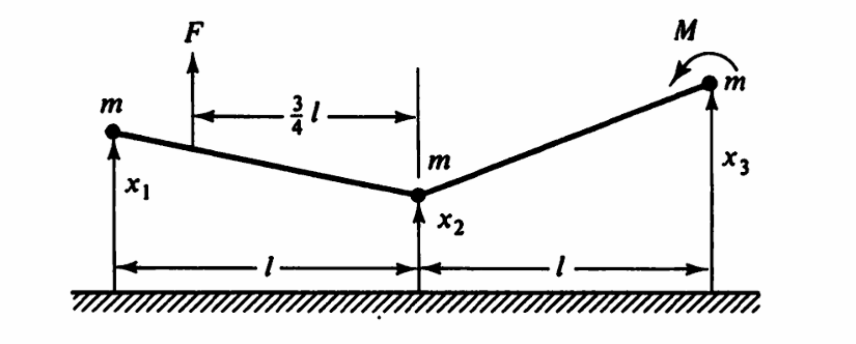


Fig 1-14 A system with an applied force and moment

The Configuration of the System is given by the co-ordinate (x_1, x_2, x_3) as

$$x_1 = q_1 + q_2 + \frac{1}{2}q_3$$
$$x_2 = q_1 - q_3$$
$$x_3 = q_1 - q_2 + \frac{1}{2}q_3$$

Find the expression the kinetic energy and find the generalized momentum?

Solution:

Given co-ordinate are:

$$x_1 = q_1 + q_2 + \frac{1}{2}q_3$$
$$x_2 = q_1 - q_3$$
$$x_3 = q_1 - q_2 + \frac{1}{2}q_3$$

Its velocity components are

$$\dot{x}_1 = \dot{q}_1 + \dot{q}_2 + \frac{1}{2}\dot{q}_3$$
$$\dot{x}_2 = \dot{q}_1 - \dot{q}_3$$
$$\dot{x}_3 = \dot{q}_1 - \dot{q}_2 + \frac{1}{2}\dot{q}_3$$
$$V^2 = \dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2$$

$$= \left(\dot{q}_1 + \dot{q}_2 + \dot{q}_{3/2}\right)^2 + (\dot{q}_1 - \dot{q}_3)^2 + \left(\dot{q}_1 + \dot{q}_2 + \frac{\dot{q}_3}{2}\right)^2$$

$$= \dot{q}_1^2 + \dot{q}_2^2 + \frac{\dot{q}_3^2}{4} + 2\dot{q}_1\dot{q}_2 + \dot{q}_1\dot{q}_3 + \dot{q}_2\dot{q}_3 + \dot{q}_1^2 + \dot{q}_3^2 - 2\dot{q}_1\dot{q}_3 + \dot{q}_1^2 + \dot{q}_2^2 + \frac{\dot{q}_3^2}{4} - 2\dot{q}_1\dot{q}_2$$
$$+ \dot{q}_1\dot{q}_3 - \dot{q}_2\dot{q}_3$$

$$V^2 = 3\dot{q}_1^2 + 2\dot{q}_2^2 + \frac{3}{2}\dot{q}_3^2$$

K.E $T = \frac{1}{2}mv^2$

$$T = \frac{1}{2}m\left[3\dot{q}_1^2 + 2\dot{q}_2^2 + \frac{3}{2}\dot{q}_3^2\right]$$

Hence the required generalized momentum is,

$$\beta_1 = \frac{\partial T}{\partial \dot{q}_1} = 3m\dot{q}_1$$

$$p_2 = \frac{\partial T}{\partial \dot{q}} = 2m\dot{q}_2$$

$$p_3 = \frac{\partial T}{\partial \dot{q}_3} = \frac{3}{2}\ \mathrm{m}\dot{q}_3$$

Example: 2

A uniform rod of mass m and length l. is constraint move in the xy-plane with end $+A$ remaining on the axis. using co-ordinate (x, θ) as the generalized co-ordinate. Find the expression of kinetic energy and Generalized momentum. P_a.

Solution:

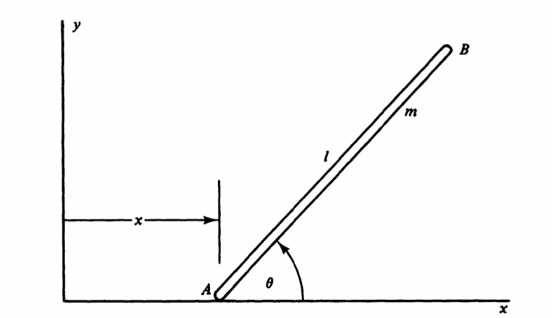


Fig 1-15

Let $G(x, y)$ be the centre of mass of the rod AB. If length l.

$$x = x + \frac{l}{2}\cos\theta \quad y = \frac{l}{2}\sin\theta$$

$$\dot{x} = \dot{x} - \frac{l}{2}\sin\theta\dot{\theta}\Big| \quad \dot{y} = \frac{l}{2}\cos\theta\dot{\theta}$$

$$v^2 = \dot{x}^2 + \dot{y}^2$$

$$= \left(\dot{x} - \frac{l}{2}\sin\theta\dot{\theta}\right)^2 + \frac{l^2}{4}\cos^2\theta\dot{\theta}^2$$

$$= \dot{x}^2 - l\,\dot{x}\,\sin\theta\,\dot{\theta} + \frac{l^2}{4}\sin^2\theta\dot{\theta}^2 + \frac{l^2}{4}\cos^2\theta\dot{\theta}^2$$

$$v^2 = \dot{x}^2 - l\dot{x}\sin\theta\dot{\theta} + \frac{l^2}{4}(\dot{\theta})^2\ [\because \cos^2\theta + \sin^2\theta = 1]$$

Kinetic Energy $T = \frac{1}{2}mv^2$

$$T = \frac{1}{2}m\left[\dot{x}^2 - l\dot{x}\dot{\theta}\sin\theta + \frac{l^2}{4}\dot{\theta}^2\right]$$

Hence the required generalized momentum is,

$$P_x = \frac{\partial T}{\partial \dot{x}} = m\dot{x} - m\frac{l}{2}\dot{\theta}\sin\theta$$

$$P_\theta = \frac{\partial T}{\partial \dot{\theta}} = -\frac{1}{2}ml\dot{x}\sin\theta + \frac{ml^2}{4}\dot{\theta}$$

Example: 3

A particle of mass m can slide without friction on a fixed circular wire of radius r. which lie in vertical plane. Using D' Alembert's principle and equation of constraints. Show that $y\ddot{x} - x\ddot{y} - gx = 0$

Proof:

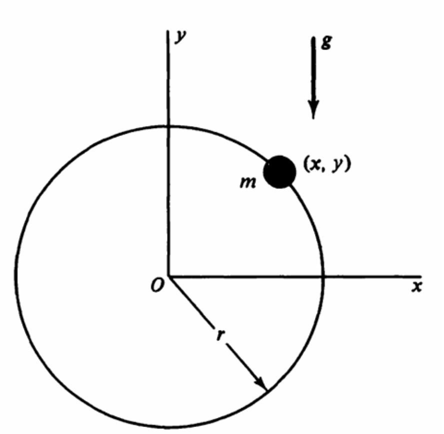


Fig 1-16

Consider a particle of mass m slide without friction on a fixed wire on a radius r.

Let r be reaction of mass m at P in time t.

The equation of motion along of x and y axis of

$m\ddot{x} = R\cos\theta$ ……….. (1)

$m\ddot{y} = R\sin\theta - mg$ … … … … (2)

Eliminate θ between (1) & (2)

From figure,

$$x = r\cos\theta \Rightarrow \cos\theta = \frac{x}{r}$$

$$y = r\sin\theta \Rightarrow \sin\theta = \frac{y}{r}$$

Sub in (1) & (2) $\Rightarrow m\ddot{x} = R\frac{x}{r}$ ………(3)

$m\ddot{y} = \frac{Ry}{r} - mg$ … … … (4)

From (3) $\Rightarrow R = \frac{mr\ddot{x}}{x}$

Sub R in (4) $\Rightarrow m\ddot{y} = \frac{y}{r}\frac{mr\ddot{x}}{x} - mg$

multiply by $\frac{x}{m} \Rightarrow x\ddot{y} = y\ddot{x} - gx = 0$

$\Rightarrow y\ddot{x} - x\ddot{y} - gx = 0$

Example: 4

A particle A of mass 2m and particle B of mass m are connected by a massless rod of length l, particle A is constrained to move along the horizontal x axis while particle B can move only along the vertical y axis. What is the equation of constraint relating x and y?

use D' Alembert's principle to obtain the equation of motion $2\ddot{x}y - x\ddot{y} - gx = 0$.

Proof:

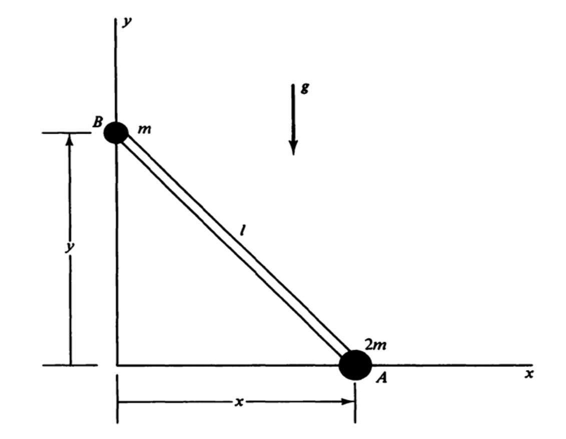


Fig 1-17

Equation of motion along, x axis. $2m\ddot{x} = s$ ... ... (1)

Equation of motion along y-axis.

$m\ddot{y} = R - mg$ ... ... (2)

Taking the momentum of force about O

$R_x = Sy$ ... ... (3)

Sub (1) in (3)

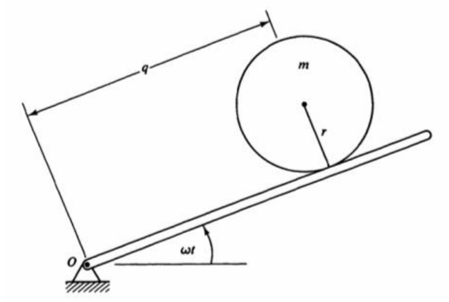
$Rx = 2m\ddot{x}y$

$R = \frac{2m\ddot{x}}{x}y$

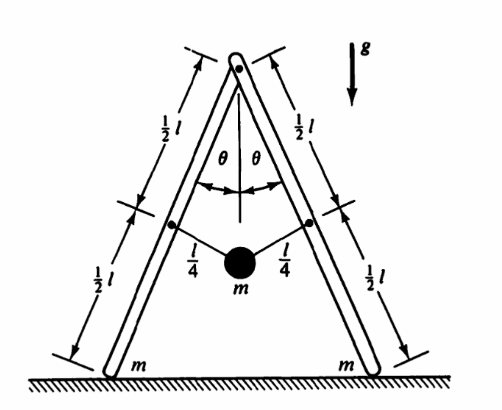
Sub in (2) $\Rightarrow m\ddot{y} = \frac{2m\ddot{x}y}{x} - mg\ (x)\frac{x}{m} \Rightarrow x\ddot{y} = 2\ddot{x}y - gx \Rightarrow 2\ddot{x}y - x\ddot{y} - gx = 0$

Exercises:

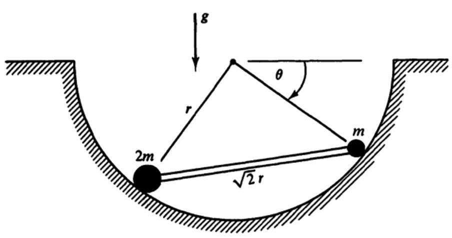
1. A disk of radius r and mass m can roll without slipping on a thin rod which rotates about a fixed point O at a constant rate ω. Obtain an expression of the form T (q,$\dot{q}$) for the total kinetic energy of the disk.



2. Two thin rods, each of mass m and length l, are pinned together at their upper ends. A particle of mass m is suspended by massless strings connected to the midpoints of the rods, as shown. Assume planar motion and use the method of virtual work to find the position of static equilibrium in the interval $0 < \theta < \frac{\pi}{6}$. Is it stable?



3. Two particles having masses m and 2m are connected by a massless rod to form a dumbbell. It can slide without friction in a circular bowl of radius r. Consider a virtual displacement $\delta\theta$ and use the principle of virtual work to obtain the value of θ at the position of static equilibrium.



UNIT II

Lagrange's Equations: Derivation of Lagrange's Equations -Examples- Integrals of the motion.

Chapter 2: Sections 2.1 to 2.3

Lagrange's Equations

There are two approaches of the subject classical dynamics. They are vectorial dynamics and Analytic dynamics.

Vectorial Dynamics:

Vectorial dynamics is based on a direct application of Newton's Law of motion. It Concentrates on the forces and motions associated with the individual parts of the system and on the interactions among these parts.

Analytical Dynamics:

Analytical Dynamics is concerned with the system as a whole and uses descriptive the scalar function. Such as kinetic and potential energies.

2.1. Derivation of Lagrange's Equation:

Express kinetic energy in terms of $q's, \dot{q}'s$ and t

proof:

Consider a system of N particle with Cartesian coordinate $x_1, x_2 \dots \cdots x_{3\mathrm{N}}$.

The total kinetic energy of a system is found from the equation

$$T = \frac{1}{2}\sum_{i=1}^{3N} m_i \dot{x}_i^2 \qquad \dots\dots\dots (1)$$

Let $q_1, q_2 \dots q_n$ be the generalized co-ordinates using the transformation equation $x's$ ass functions $q's$ and time t.

Therefore x_{k}=x_{k} (q,t) , here k=1,2,…..3N ………..(2)

$\therefore x_k = x_k(q_1, q_2 - \cdots q_n, t)$

$$\therefore \dot{x}_k = \frac{dx_k}{dt} = \frac{\partial x_k}{\partial q_1}\ddot{q}_1 + \frac{\partial x_k}{\partial q_2}\dot{q}_2 + \cdots + \frac{\partial x_k}{\partial q_k}\dot{q}_n + \frac{\partial x_k}{\partial t}dt$$

$$\dot{x}_k = \sum_{i=1}^{n} \frac{\partial x_k}{\partial q_i}\dot{q}_i + \frac{\partial x_k}{\partial t}dt \qquad \ldots\ldots\ldots\ldots..(3)$$

Here $\dot{x}_k$ is linear in $\dot{q}'s$ and $\frac{\partial x_k}{\partial q_i}, \frac{\partial x_k}{\partial t}$ are functions of $q's$ and t.

Sub (3) in (1) we get $T(q, \dot{q}, t) = \frac{1}{2}\sum_{i=1}^{3N} m_k \left[\sum_{i=1}^{k} \frac{\partial x_k}{\partial q_i}\dot{q}_i + \frac{\partial x_k}{\partial t}dt\right]^2$

Let us group the terms according to their degree in using $\dot{q}'s$ the notation

$T = T_2 + T_1 + T_0$ ………….. (5)

Note:

(1)We have $T = T_2 + T_1 + T_0$

where T_2 is a homogenous quadratic function of $\dot{q}_i's. T_1$ is a homogenous linear function of $\dot{q}_i's$ *and* T_0 includes the remaining terms which are function of $q's$ ant $t's$.

Here T_2 is of the form

$T_2 = \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij}\dot{q}_i\dot{q}_j$ …………….. (6)

Where $m_{ij} = m_{ji} = \sum_{i=1}^{3N} m_k \frac{\partial x_k}{\partial q_i}\frac{\partial x_k}{\partial q_j}$ …………(7)

Also $T_1 = \sum_{i=1}^{n} a_i\dot{q}_i$ ……….. (8)

where $a_i = \sum_{i=1}^{3N} m_k \frac{\partial x_k}{\partial q_i}\frac{\partial x_k}{\partial q_j}$ ………..(9)

$T_0 = \frac{1}{2}\sum_{i=1}^{3N} m_k (\frac{\partial x_k}{\partial q_t})^2$ ………… (10)

Assuming $m_k > 0$ for all k, then the total kinetic energy T is a positive definite quadratic function of $\dot{x}_i$.

i.e., T is zero if all $\dot{x}_i' s$ are zero. If any of the $x's$ is non-zero the kinetic energy is positive.

(2) For any real system the kinetic energy is zero only if the system is motionless otherwise it is positive.

(3) T_2 must be a positive definite quadratic functions of $\dot{q}_i' s$. The positive definite nature of T_2 restricts the possible values of the inertia coefficients m_{ij}.

Consider the symmetric n$\times n$ generalized inertia matrix m, the necessary and sufficient and that be a positive definite are that $m_{11} > 0$, $\begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix} > 0$ $\begin{vmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & \ddots & \vdots \\ m_{n1} & \cdots & m_{nn} \end{vmatrix} > 0$

This is equivalent to the determinant of the matrix and all principal minors be positive. Also all the inertia coefficients along the main diagonal must be positive.

(4) From equation (9) & (10) T_1 & T_{10} are non-zero only for the case of Rheonomic system. It follows that the kinetic energy T of a scleronomic system is a homogenous quadratic functions of from equation (7) the inertia coefficients m_{ij} are functions of $q's$ but not of time

Obtain Lagrange's equation for holonomic system:

Proof:

Part (i): Lagrange's from of D'Alemberts

Consider a system of N particle with Cartesian co-ordinate with $(x_1, x_2 \dots x_{3N})$ by

D ' Alembert's principle. $\sum_{k=1}^{3N} \left(\overrightarrow{F_k} - m_k \ddot{\overrightarrow{x_k}}\right)\delta\overrightarrow{x_k} = 0$ ……….. (1)

Where F_k is the applied force along the component associate with x_K.

∴ The virtual displacement δx_k can be expressed in terms of δq's as follows:

$x_k = x_k(q_1, q_2 \dots q_k, t)$ …………… (2)

$$\delta x_k = \frac{\partial x_k}{\partial q_1}\delta q_1 + \frac{\partial x_k}{\partial q_2}\delta q_2 + \cdots \dots . + \frac{\partial x_k}{\partial q_n}\delta q_n$$

$$\therefore \delta x_k = \sum_{i=1}^{n} \frac{\partial x_k}{\partial q_i} \delta q_i \text{ For } k = 1,2 \ldots 3n$$

Sub in (1) $\sum_{i=1}^{3N} \sum_{i=1}^{n} \left[F_k \frac{\partial x_k}{\partial q_i} - m_k \overrightarrow{\ddot{x}_k} \frac{\partial x_k}{\partial q_i}\right] \delta q_i = 0$ …………… (3)

Now, differentiation equation (2) with respect to t,

$\dot{x}_k = \sum_{j=1}^{n} \frac{\partial x_k}{\partial q_j} \dot{q}_j + \frac{\partial x_k}{\partial t_i}$ ……… (4)

(i.e.,) Diff $\dot{x}_k$ partially w.r.t $\dot{q}_i$

$\frac{\partial \dot{x}_k}{\partial \dot{q}_i} = \frac{\partial x_k}{\partial q_i}$ ……….(*)

Again Diff (4) w.r.t q_i. $\frac{\partial \dot{x}_k}{\partial q_i} = \sum_{j=1}^{n} \frac{\partial x_k}{\partial q_i \partial q_j} \dot{q}_j + \frac{\partial x_k}{\partial q_i \partial t}$ ………… (5)

By changing the order of differentiation, we get

$\frac{d}{dt}\left(\frac{\partial x_k}{\partial q_i}\right) = \sum_{j=1}^{n} \frac{\partial x_k}{\partial q_i \partial q_i} \dot{q}_j + \frac{\partial x_k}{\partial t \, \partial q_i}$ ……….. (6)

from (5) & (6)

$\frac{d}{dt}\left(\frac{\partial x_k}{\partial q_i}\right) = \frac{\partial \dot{x}_k}{\partial q_i}$ ……….(**)

$$\therefore K \cdot E,\ T = \frac{1}{2} \sum_{k=1}^{3N} m_k \dot{x}_k{}^2.$$

We can write the generalized momentum P_i is the form $P_i = \frac{\partial T}{\partial \dot{q}_i}$

Now diff ‘ T’ partially w.r.to $\dot{q}_i$

$$\frac{\partial T}{\partial \dot{q}_i} = \frac{1}{2}\sum_{k=1}^{3N} m_k 2\vec{\dot{x}_k}\frac{\partial \dot{x}_k}{\partial \dot{q}_i}$$

$$\frac{\partial T}{\partial \dot{q}_i} = \sum_{k=1}^{3N} m_k \dot{\vec{x}}_k \frac{\partial \vec{x}_k}{\partial q_i} \quad \text{by } (*)$$

$Now\ , again\ diff\ w.r.to\ 't'$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = \sum_{k=1}^{3N} m_k \vec{\ddot{x}_k}\frac{\partial \vec{x_k}}{\partial q_i} + \sum_{k=1}^{3N} m_k \vec{\dot{x}_k}\frac{d}{dt}\left(\frac{\partial \dot{x}_k}{\partial q_i}\right) \quad \ldots\ldots\ldots..(7)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = \sum_{k=1}^{3N} m_k \vec{\ddot{x}_k}\frac{\partial \vec{x}_k}{\partial q_i} + \sum_{k=1}^{3N} m_k \vec{\dot{x}_k}\frac{\partial \dot{x}_k}{\partial q_i} \quad \ldots\ldots\ldots\ldots\ldots(8)$$

Now diff T partially w.r.t q_i

$\frac{\partial T}{\partial q_i} = \frac{1}{2}\sum_{k=1}^{3N} m_k 2\dot{\vec{x}}_k \frac{\partial \dot{x}_k}{\partial q_i}$ .... (9)

Sub (9) in (8).

$$(8) \Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = \sum_{k=1}^{3N} m_k \vec{\ddot{x}_k}\frac{\partial \vec{x}_k}{\partial q_i} + \frac{\partial T}{\partial q_i}$$

Now the generalized force $\vec{Q}_i = \sum_{k=1}^{3N} \vec{F}_k \frac{\partial \vec{x}_k}{\partial q_i}$ for $i = 1 \ldots 3n$ ....(11)

Sub (10), (11) in (3)

$(3) \Rightarrow \sum_{i=1}^{n} \left[\vec{Q}_i - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) + \frac{\partial T}{\partial q_i}\right]\delta q_i = 0$ ....(12)

∴ This is known as Lagrange’s form of D' Alembert's principle in terms of generalized co-ordinates.

Part (ii):

To obtain the Lagrange’s equation of Motion:

In order to neglect virtual work of the constraint force, we made the restriction forces on δq that they must conform to the instantaneous constraints.

Let us make the additional assumption that the system is holonomic and its configuration described by a set of independent generalized co-ordinates.

If they $\partial q_i's$ are independent, then the coefficient of each in equation (12) must be zero.

$\therefore\ \vec{Q}_i - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) + \frac{\partial T}{\partial \dot{q}_i} = 0$ (or)

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial \dot{q}_i} = \vec{Q}_i \quad i = 1,2 \ldots N$$

This n-equations known as Lagrange's equation of motion

Part (iii)

To obtain the standard form of Lagrange's equation for holonomic system.

Let us make additional assumption that all the generalized forces are derivable from a potential function $V = V(q, t)$.

$\therefore Q_i = -\frac{\partial v}{\partial q_i}$

Sub in (13), we get

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = -\frac{\partial v}{\partial q_i}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial (T-v)}{\partial q_i} = 0 \qquad \ldots\ldots\ldots\ldots (14)$$

$\therefore$ The Lagrange's function $L =$L(q, $\dot{q}, t)$ is $L = T - V$.

$$\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial v}{\partial \dot{q}_i}$$

$$\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} - 0 \quad (\, v = v(q, t))$$

$(14) \Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0.\ \ i = 1$ to 3

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

Which is the Standard form of the Lagrange's Equation.

Another form of Lagrange's Equation:

Another form of Lagrange equation can be written for systems in which the generalized forces are not wholly derivable from a potential function.

Let $Q_i = -\frac{\partial v}{\partial q_i} + Q_i{}'$ ………… (1)

We have Lagrange equation of motion is $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i$ i=1,2…n ……….(2)

Sub (1) in (2), we get

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = -\frac{\partial v}{\partial q_i} + Q_i{}' \quad \text{i=1,2…n}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial}{\partial q_i}(T - v) = Q_i{}'$$

We have $L = T - V$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial T}{\partial q_i}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i{}' \quad \text{i=1,2…n}$$

Where $Q_i{}'$ are those generalized forces not derivable from a potential function

Application of Lagrange's Equation:

The application Lagrange equations in the standard form is the form of equations of motion.

Since the generalized momentum is linear in $\dot{q}_i{}'s$

$P_i = \frac{\partial T}{\partial q_i} = \sum_{j=1}^{n} m_{ij}\ \dot{a}_j + a_i$ ………… (1)

Where m_{ij} and a_i are functions of $\dot{q}_i's$ and t.

Hence the equation of motion are linear in $\ddot{q}_i's$

Since all the terms containing $\ddot{q}_i's$ arise from differentiating equation (1) with respect to t

Now diff equation (1) w.r.to 't', we get

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = \sum_{j=1}^{n} m_{ij}\ \ddot{q}_j + \sum_{j=1}^{n} m_{ij}\ \dot{q}_j + a_i \quad \ldots\ldots\ldots\ldots\ldots (2)$$

Where $m_{ij} = \sum_{l=1}^{n} \frac{\partial m_{ij}}{\partial q_i} \dot{q}_j + \frac{\partial m_{ij}}{\partial \text{t}}$ …………(3)

$$\dot{a}_i = \sum_{j=1}^{n} \frac{\partial a_i}{\partial q_i} \dot{q}_j + \frac{\partial a_i}{\partial \text{t}} \quad \ldots\ldots\ldots\ldots (4)$$

Now $\sum_{j=1}^{n} m_{ij}\, \dot{q}_j = \sum_{j=1}^{n} \sum_{l=1}^{n} \frac{\partial m_{ij}}{\partial q_i} \dot{q}_i \dot{q}_j + \sum_{j=1}^{n} \frac{\partial m_{ij}}{\partial \text{t}} \dot{q}_j$

$$= \frac{1}{2} \sum_{j=1}^{n} \sum_{l=1}^{n} \left(\frac{\partial m_{ij}}{\partial q_l} + \frac{\partial m_{ij}}{\partial q_i}\right) \dot{q}_i \dot{q}_l + \sum_{j=1}^{n} \frac{\partial m_{ij}}{\partial \text{t}} \dot{q}_j \quad \ldots\ldots\ldots (5)$$

$$\frac{\partial T_2}{\partial q_i} = \sum_{j=1}^{n} \sum_{l=1}^{n} \frac{\partial m_{il}}{\partial q_i} \dot{q}_i \dot{q}_j \quad \ldots\ldots\ldots\ldots (6)$$

$$\frac{\partial T_1}{\partial q_i} = \sum_{j=1}^{n} \frac{\partial a_j}{\partial q_i} \dot{q}_j \quad \ldots\ldots\ldots\ldots (7)$$

Here several dummy indices have been changed.

Sub the equation (2), (3),(4),(5),(6),(7) in Lagrange equation, we obtain

$$\sum_{j=1}^{n} m_{ij}\, \ddot{q}_j + \frac{1}{2} \sum_{j=1}^{n} \sum_{l=1}^{n} \left(\frac{\partial m_{ij}}{\partial q_l} + \frac{\partial m_{ij}}{\partial q_i} - \frac{\partial m_{jl}}{\partial q_i}\right) \dot{q}_j \dot{q}_l + \left(\sum_{j=1}^{n} \frac{\partial m_{ij}}{\partial \text{t}} + \frac{\partial a_i}{\partial q_j} - \frac{\partial a_j}{\partial q_i}\right) \dot{q}_j + \frac{\partial a_i}{\partial \text{t}} - \frac{\partial T_0}{\partial q_i} + \frac{\partial \text{v}}{\partial q_i} = 0 \qquad \text{i=1, 2 ,…..,n} \quad \ldots\ldots\ldots\ldots..(8)$$

This notation can be shortened by using Christoffel symbol of the first kind which is applied for the quadratic function from T_2.

Let $[jl, i] = \frac{1}{2}\left(\frac{\partial m_{ij}}{\partial q_l} + \frac{\partial m_{ij}}{\partial q_i} - \frac{\partial m_{jl}}{\partial q_i}\right)$ …………(9)

Then equation (8) can be written as

$$\sum_{j=1}^{n} m_{ij}\,\ddot{q}_j + \sum_{j=1}^{n}\sum_{l=1}^{n}[jl,i]\,\dot{q}_j\dot{q}_l + \sum_{j=1}^{n}\gamma_{ij}\,\dot{q}_j + \sum_{j=1}^{n}\frac{\partial m_{ij}}{\partial \text{t}}\dot{q}_j + \frac{\partial a_i}{\partial \text{t}} - \frac{\partial T_0}{\partial q_i} + \frac{\partial \text{v}}{\partial q_i} = 0$$

i=1, 2, …… n ………… (10)

Where γ_{ij} is an element of a skew symmetric matrix and is given by

$\gamma_{ij} = -\gamma_{ij} = \frac{\partial a_i}{\partial q_j} - \frac{\partial a_j}{\partial q_i}$ the n-equations of (8) or (10) are the equations of motion.

Obtain Lagrange's equation for non-holonomic system.

Proof:

Let us consider a system of N - particle. Where configuration is described by n generalized co-ordinates. For a non-holonomic system. There must be move generalized co-ordinates than the number of degree of freedom.

Therefore, the $\delta q's$ are no longer independent if we assume a virtual displacement consistent with the constraints. Consider m non-holonomic constraint equation of the form.

$\sum_{i=1}^{n} a_{ji}dq_i + a_{ji}dt = 0, j = 1, \dots m$ ………… (1)

δq satisfied the condition

$$\sum_{i=1}^{n} a_{ji}\delta q_i = 0, j = 1, \dots m \dots (2)$$

Let us assume that the generalized applied force Q_i is obtained from potential function V.

$$Q_i = -\frac{\partial v}{\partial q_i}\; i = 1 \dots n$$

Also the constraints are assumed to be workless. So, that generalized constraint force C_i must be meet the condition $\sum_{i=1}^{n} c_i\delta q_i = 0$ …….(3)

For any virtual displacement is consistent with the constraints.

By Lagrange's multiplier method. We obtain m equations by multiply equation (2) by λ_j

$\Rightarrow (2) \times \lambda_j \Rightarrow \lambda_j \sum_{i=1}^{n} a_{ji}\delta q_i = 0, \quad j = 1, \dots m$

$\sum_{j=1}^{m} \lambda_j \sum_{i=1}^{n} a_{ji}\delta q_i = 0$ ………… (4)

Now equation (3) −(4) $\Rightarrow \sum_{i=1}^{n}\left[c_i - \sum_{j=1}^{m} a_{ji}\lambda_j\right]\delta q_i = 0$ ………..(5)

Let us choose $\lambda' s$ Such that $c_i = \sum_{j=1}^{m} a_{ji}\lambda_j$ ………(6) where i=1,2,…..n.

Then the co-efficient of $\delta q' s$ are zero in equation (5)

In other words δq can be chosen independently.

We know that Lagrange's equation can be written for system S in which the generalized force are not fully derivable from potential function is $Q_i = -\frac{\partial v}{\partial q_i} + Q_i$ ……..(7)

Where Q_i' are those generalized force not derived from a potential function

In this case the Lagrange's equation of $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i'$ ……….(8)

Also $\delta q' s$ can be chosen independently with these assumption we can equate the generalized force C_i with Q_i' using equations (6) and (8) we get

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \sum_{j=1}^{m} \lambda_j a_{ji} \; i = 1 \dots n$$

This is known as standard form of Lagrange equation is non-holonomic System.

2.2. Examples:

Example:1

Find the differential equations of motion for a spherical pendulum of length l.

Solution:

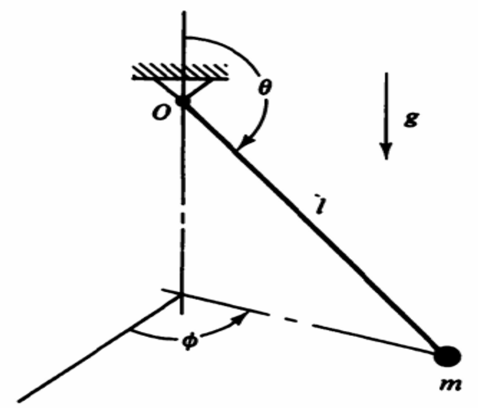


Fig. 2-1. A spherical pendulum.

Let $op = l$ be the length of spherical pendulum.

Let $P(l, \theta, \phi)$ be a position of P of mass m at time ' $\pm$ '.

The velocity components of P are $(\dot{\gamma}, r\dot{\theta}, r\sin\theta\dot{\phi})$ here $r = l$.

The velocity components are $(0, l\dot{\theta}, l\sin\theta\dot{\phi})$ here $r = l$.

$$\therefore v^2 = 0^2 + l^2\dot{\theta}^2 + l^2\sin^2\theta\dot{\phi}^2$$
$$v^2 = l^2[\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2]$$
$$K \cdot E = T = \frac{1}{2}mvv^2$$
$$T = \frac{1}{2}ml^2[\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2]$$

potential energy $v = -mgh\cos(180 - \theta)$ $\quad$ $[\cos(180 - \theta) = -\cos\theta]$

$v = mgh\cos\theta$

$\therefore$ The Lagrangian function is,

$$L = T - V$$
$$L = \frac{1}{2}ml^2[\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2] - mgh\, cos\theta$$

$\therefore$ The required equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

For $q_i = \theta, \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$

$$\frac{d}{dt}[\frac{1}{2}\left(ml^2\dot{\theta}\right) - \left(\frac{1}{2}ml^2\left(\dot{\phi}^2\, 2\sin\theta\, cos\,\theta\right) - mgh\sin\theta\right) = 0$$

$$\div\, ml^2 \quad \frac{d}{dt}\left(\dot{\theta}\right) - \dot{\phi}^2\ \sin\theta\, cos\,\theta \text{-} \frac{gh}{l^2}\sin\theta = 0$$

$\ddot{\theta} - \sin\theta\cos\theta\,\dot{\phi}^2 - \frac{gh}{l^2}\sin\theta = 0$ ………….. (1)

Now, the Lagrangian equation of motion for $q_2 = \phi$ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0$$
$$\frac{d}{dt}\left(\frac{1}{2}ml^2[\sin^2\theta \times 2\dot{\phi}\right) - 0 = 0$$
$$\div\, ml^2\ \frac{d}{dt}\left(\sin^2\theta\dot{\phi}\right) = 0$$

$\sin^2\theta\ \ddot{\phi} + \dot{\phi}2\, sin\theta\cos\theta\,\dot{\theta} = 0$ ………… (2)

Equation (1) & (2) are required equation of motion.

Note:

The ϕ equation of motion is immediately integrable in the above example because $\frac{\partial L}{\partial \phi} = 0$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - 0 = 0$$
$$\frac{d}{dt}\left(P_\phi\right) = 0$$

An angular momentum of $P_{\phi} =$ constatn c.

The angular momentum of spherical pendulum in the vertical direction is constant.

Example 2:

Find the equation of motion for the double pendulum. (or) A double pendulum consists of two particles suspended by the massless rod as shown in figure. Assuming that all motion takes place in a vertical plane. Find the i) Differential equation of motion ii) Linearized the equation assuming the small motion

Proof:

To find Kinetic Energy, the absolute velocity of lower particle is equal to vector sum of.

(i) The absolute velocity of the upper particle.

(ii) The velocity of the h_1 lower particle relative to the upper particle.

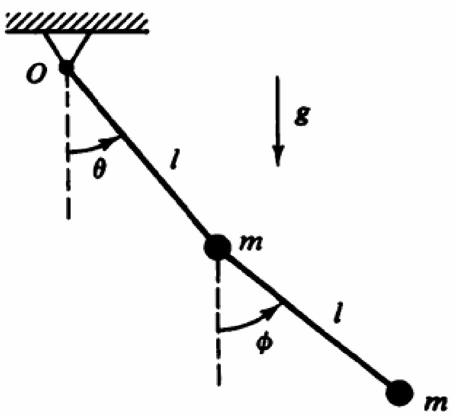


Fig. 2-2. A double pendulum.

Since the two velocities differ in the angle $(\phi - \theta)$. We have,

$\therefore V^2 = l^2\left[2\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\cos(\phi - \theta)\right]$

$\therefore K \cdot E\ T = \frac{1}{2}mv^2$

$= \frac{1}{2}ml^2\left[2\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\cos(\phi - \theta)\right]$

$P \cdot E \Rightarrow \vec{V} = -mg(h_1 + h_2)$

$= -mg(2l\cos\theta + l\cos\phi)$

$V = -mgl(2\cos\theta + \cos\phi)$

$\therefore$ The Lagrangian function,

$L = T - V$

$L = \frac{1}{2}ml^2\left[2\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\cos(\phi - \theta)\right]$

$+ mgl[2\cos\theta + \cos\phi]$

$\therefore$ The Lagrangian equation of motion are $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0$

$\frac{d}{dt}\left[\frac{1}{2}ml^2\left[2\,.2\theta\ + 2\dot{\phi}\cos(\phi - \theta)\right]\right.$- $\frac{1}{2}ml^2\ 2\ \dot{\theta}\dot{\phi}\sin(\phi - \theta)$(-1)+ $mgl(-\sin\theta) = 0$

$\frac{d}{dt}\left[ml^2\left[2\ \dot{\theta}\ + \dot{\phi}\cos(\phi - \theta)\right]\right]$- $ml^2\ \dot{\theta}\dot{\phi}\sin(\phi - \theta)$+2 $mgl\sin\theta = 0$

$ml^2[2\ddot{\theta} + \dot{\phi}\left(-\sin(\phi - \theta)\left(\dot{\phi} - \dot{\theta}\right) + \cos(\phi - \theta)\ddot{\phi}\right] - ml^2\dot{\theta}\dot{\phi}\sin(\phi - \theta)$

$+2mgl\sin\theta = 0$

$\div ml^2,\ 2\ddot{\theta} - \dot{\phi}^{\,2}\sin(\phi - \theta) + \dot{\theta}\dot{\phi}\ \sin(\phi - \theta) + \ddot{\phi}\cos(\phi - \theta) - \dot{\theta}\dot{\phi}\ \sin(\phi - \theta)$

$+\frac{2g}{l}\sin\theta = 0$

$2\ddot{\theta} - \dot{\phi}^{\,2}\sin(\phi - \theta) + \ddot{\phi}\cos(\phi - \theta) + \frac{2g}{l}\sin\theta = 0$ …………(1)

The Lagrangian equation of motion for $q_2 = \phi$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0.$$

$$\frac{d}{dt}\left(ml^2\dot{\phi} + ml^2\dot{\theta}\cos(\phi - \theta)\right) - \left[-ml^2\dot{\theta}\dot{\phi}\sin(\phi - \theta)\dot{\phi} - mgl\sin\phi\right] = 0$$

$$\div\, ml^2 \;\; \Rightarrow \ddot{\phi} + \ddot{\theta}\cos(\phi - \theta) - \dot{\theta}\sin(\phi - \theta)\left(\dot{\phi} - \dot{\theta}\right) + \dot{\theta}\dot{\phi}\sin(\phi - \theta)\dot{\phi} + \frac{g}{l}\sin\phi = 0 \qquad \dots\dots\dots\dots\dots (2)$$

For linearity of small motion

Put $\phi - \theta \simeq 0.$

Sub in (1) & (2)

we get,

$$2\ddot{\theta} + \ddot{\phi} + \frac{2g}{l}\sin\theta = 0. \;\; \& \; \ddot{\phi} + \ddot{\theta} + \frac{g}{l}\sin\phi = 0$$

are the equation of motion.

For θ is small, $\sin\theta \cong \theta$

$$2\ddot{\theta} + \ddot{\phi} + \frac{2g}{l}\theta = 0 \;\; \& \;\; \ddot{\phi} + \ddot{\theta} + \frac{g}{l}\phi = 0.$$

Example 3:

A block of mass m_2 can slide on another block of mass m_1 which in turn slide on horizontal surface in x_1 direction. Using $x_1 \& x_2$ as the co-ordinate obtain the diff eq of motion.

Solution:

Let x_1 be the absolute re at displacement of particle of mass m_1.

Let x_2 be the displacement of the block of mass m_2 relative to mass m_1.

Let v_1 be the velocity of mass $m_1 \therefore \dot{v}_1 = \dot{x}_1$

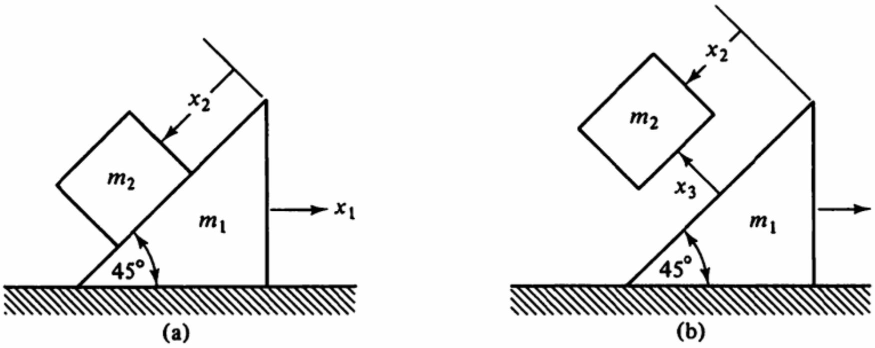


Fig. 2-3. A system of sliding blocks.

To find velocity of mass $m_2(v_2)$:

By using cosine formula,

$$[a^2 = b^2 + c^2 - 2bc\cos A]$$
$$v_2^2 = \dot{x}_1^2 + \dot{x}_2^2 - 2\dot{x}_1\dot{x}_2\cos 45^\circ$$
$$= \frac{1}{2}[m_1v_1^2 + m_2v_2^2]$$
$$T = \frac{1}{2}\left[m_1\dot{x}_1^2 + m_2\left(\dot{x}_1^2 + \dot{x}_2^2 - \frac{2\dot{x}_1\dot{x}_2}{\sqrt{2}}\right)\right]$$
$$P \cdot EV = mgh.$$

$$= m_2gx_2\cos 135^\circ$$
$$= m_2gx_2\sin 45^\circ$$
$$= -m_2\frac{gx_2}{\sqrt{2}}$$

∴ The Lagrangian function $L = T - V$.

$$L = \frac{1}{2}[(m_1 + m_2)\dot{x}_1^2 + m_2\dot{x}_2^2 - \sqrt{2}m_2\dot{x}_1\dot{x}_2] + \frac{m_2}{\sqrt{2}}gx_2$$

∴ The Lagrangian eq of motion for $q_1 = x_1$ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0$$
$$\frac{d}{dt}\left[(m_1 + m_2)\dot{x}_1 - \frac{1}{\sqrt{2}}m_2\dot{x}_2\right] - 0 = 0$$
$$(m_1 + m_2)\ddot{x}_1 - \frac{m_2}{\sqrt{2}}\ddot{x}_2 = 0$$

The Lagrangian equation of motion for $q_2 = x_2$ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0$$

$$\frac{d}{dt}\left(m_2\dot{x}_2 - \frac{1}{\sqrt{2}}m_2\dot{x}_1\right) - \frac{m_2 g}{\sqrt{2}} = 0.$$

$$m_2\ddot{x}_2 - \frac{m_2}{\sqrt{2}}\ddot{x}_1 - \frac{m_2 g}{\sqrt{2}} = 0.$$

$$multiply\ by\ \frac{\sqrt{2}}{m_2} \Rightarrow \sqrt{2}\ddot{x}_2 - \ddot{x}_1 - g = 0$$

Example 4:

A particle of mass ' m ' can slide without. Friction on the inside of ' q ' which is a bend. in the form of p circle of radius ' r '. The tube rotated above a vertical diameter with a constant angular velocity w as shown in the figure. Write the differentiation equation of motion.

Solution:

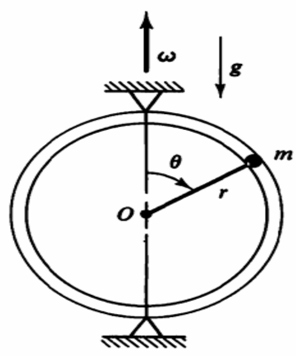


Fig. 2-4. A particle in a whirling tube.

Let p be the position of particle of mass m slide, inside a tube (circle)

Let $\vec{V} = r\dot{\theta}\vec{\imath} + r\omega\sin\theta\vec{\jmath}$

be the velocity at P, at time ' t '

$\therefore V^2 = r^2\dot{\theta}^2 + r^2\omega^2 \sin^2\theta.$

$$K, ET = \frac{1}{2}mv^2$$
$$= \frac{1}{2}m\left[r^2\dot{\theta}^2 + r^2\dot{\omega}^2\sin^2\theta\right]$$
$$P\cdot EV = mgh$$
$$= mgr\cos\theta.$$

By Lagrangian function $L = T - V$

$$L = \frac{1}{2}m\left[r^2\dot{\theta}^2 + r^2\omega^2\sin^2\theta\right] - mgr\cos\theta.$$

[here r fixed].

The equation of motion for $q_1 = 0$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0.$$
$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) - \left[mr^2\omega^2\sin\theta\cos\theta + mgr\sin\theta\dot{\theta}\right]$$
$$mr^2\ddot{\theta} - mr^2\omega^2\sin\theta\cos\theta - mgr\sin\theta\dot{\theta} = 0.$$

This is the required equation of motion.

Example 5:

Obtain the diff equation of motion for a simple pendulum.

Solution:

Let M be the mass of the ball P & l be the length of simple pendulum.

Let op be the position of M when displaced through an angle θ at time t.

$\therefore$ Arc length

$$AP = s = r\theta$$
here $S = l\theta$ ($\because r = l$)
(distance)

$\therefore$ The velocity at $p = v = l\dot{\theta}$.

$$K \cdot E\ T = \frac{1}{2}mv^2$$
$$= \frac{1}{2}ml^2\dot{\theta}^2$$
$$P \cdot E\ V = mgh$$
$$= mgl(1 - \cos\theta)$$

$\therefore$ By Lagrangian function $L = T - V$

$$L = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - \cos\theta)$$

($\because$ here l fixed).

The equation of motion for $q_1 = 0$.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$
$$\frac{d}{dt}\left(ml^2\dot{\theta}\right) - mgl\sin\theta \cdot \dot{\theta} = 0$$
$$ml^2\ddot{\theta} - mgl\sin\theta \cdot \dot{\theta} = 0$$
$$\div\, ml^2,\ \ddot{\theta} - \frac{g}{l}\sin\theta \cdot \ddot{\theta} = 0.$$

2.3. Integrals of the Motion:

Definition: Integral of motion

Take a holonomic system with n independent general coordinate $q_1, q_2 \dots \dots q_n$ for which the Lagrangian equation of motion are,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, \text{ for } (i = 1,2 \dots - n)$$

There constant may be evaluated on the basis of initial condition of the system.

These $2n$ relation may be solved for $n\ q's \times n\dot{q}'s$ as a function of α and t.

$\therefore$ we get,

$$q_i = q_i(\alpha_1, \alpha_2 \dots \alpha_n, t)$$
$$\dot{q}_i = \dot{q}_i(\alpha_1, \alpha_2 \dots \alpha_n, t)$$

These $2n$ equation are called integral of motion.

Definition: Ignorable Co-ordinate.

If a co-ordinate q_i is absent in the Lagrangian function L is called ignorable co-ordinate (cyclic co-ordinate of the system).

(i.e.,) $\frac{\partial L}{\partial q_i} = 0$

∴ By equation of motion.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$
$$\frac{d}{dt}(P_i) - 0 = 0$$

Integrate, $P_i = \beta_i$ is constant.

(i.e.,) An ignorable coordinate the general momentum is constant.

Example: Kepler's problem

The problem of motion of a particle of mass I unit which is attracted by a force is inversely proportional to square of its radius.

(i.e.,) $\vec{F} \propto \frac{1}{\gamma^2}$

Let $p(r, \theta)$ be a position of particle of mass 1 unit at time t. Let $\vec{F}$ be a altractive force towards 0 .

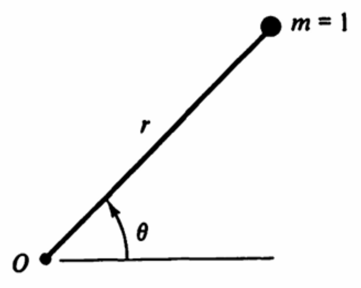


Fig. 2. 5.The Kepler problem in terms of polar coordinates

$\therefore$ The velocity at $p = V = \dot{\gamma}\vec{\imath} + r\dot{\theta}\vec{\jmath}$.

$$V^2 = \dot{\gamma}^2 + r^2\dot{\theta}^2$$
$$K.E\ \ T = \frac{1}{2}mv^2$$
$$= \frac{1}{2}\omega\left[\dot{r}^2 + \gamma^2\dot{\theta}^2)\right]\ \ [\because \text{ given } m = 1]$$
$$T = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$
$$P.E\ \ V = \int \left(\frac{\mu}{r^2}\right)dr,\ \ \therefore \vec{F} \propto \frac{1}{r^2}, \vec{F}_j = \frac{\mu}{r^2}$$
$$V = \frac{-\mu}{r}.$$

$\therefore By$ Lagrangian function $L = T - V$

$$= \frac{1}{2}\left(\dot{\gamma}^2 + \gamma^2\dot{\theta}^2\right) + \frac{\mu}{\gamma}$$

(i) The equation of motion for $q_1 = r$.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$
$$\frac{d}{dt}(\dot{r}) - \left(r\dot{\theta}^2 - \frac{\mu}{r^2}\right) = 0$$
$$\ddot{r} - r\dot{\theta}_2^2 + \frac{\mu}{r^2} = 0 \ldots\ldots\ldots\ldots (1)$$

(ii) The equation of motion for $q_2 = 0$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$
$$\frac{d}{dt}\left(r^2\dot{\theta}\right) - 0 = 0$$

Integrate, $\gamma^2\dot{\theta} = \beta$ ………..(2)

[here $\frac{\partial L}{\partial \dot{\theta}} = 0 \Rightarrow \theta$ is ignorable co-ordinate]

Eliminate $\dot{\theta}$ between (1) & (2).

$$(1) \Rightarrow \ddot{\gamma} - \gamma\left(\frac{\beta^2}{r^4}\right) + \frac{\mu}{r^2} = 0$$

$$\Rightarrow \ddot{\gamma} - \frac{\beta^2}{r^3} + \frac{\mu}{r^2} = 0 \text{ where } \beta = r^2\dot{\theta}.$$

Definition: Routhian Function *R*

Obtain the Lagrangian equation of motion for Routhian function R.

Consider a holonomic system with ' n ' general co-ordinate $q_1, q_2 \dots q_n$.

Suppose that $q_1, q_2 \dots - q_k$ are ignorable co-ordinate $[k < n]$.

$L = L(q_{k+1}, q_{k+2} \dots - q_n, \dot{q}_{k+1}, \dot{q}_{k+2} - \dot{q}_n, t)$

Now, we define Routhian function R as,

$$R = L - \sum_{i=1}^{k} \beta_i \dot{q}_i \text{ where } \beta_i = \frac{\partial L}{\partial \dot{q}_i}$$
$$R = R(q_{k+1}, \dot{q}_{k+2}, \cdots - q_n, \dot{q}_{k+1}, \dot{q}_{k+2}, \cdots \dot{q}_n, t)$$

To obtain: Lagrangian equation of motion for Routhian Function

$$L(q_{k+1}, q_{k+2}, q - q_n, \dot{q}_1, \dot{q}_2 - -\dot{q}_n - t)$$
$$R\big(q_{k+1}, q_{k+2} \dots - q_n, \dot{q}_{k+1}, \dot{q}_{k+2} \dots - \dot{q}_n, \beta_1, \beta_2 \dots .., \beta_{\mathrm{k}}, t\big) \dots \dots \dots .(1)$$

$\mathrm{R} = L - \sum_{i=1}^{k} \beta_i \dot{q}_i$ .... (2)

By (1) $\Rightarrow \delta R = \sum_{i=k+1}^{n} \frac{\partial R}{\partial q_i} \delta q_i + \sum_{i=k+1}^{n} \frac{\partial R}{\partial \dot{q}_i} \delta \dot{q}_i + \sum_{i=1}^{k} \frac{\partial R}{\partial \beta_i} \delta \beta_i + \frac{\partial R}{\partial t} \delta \mathrm{t}$ ....(3)

Also By (2), $\delta R = \delta L - \delta(\sum_{i=1}^{K} \beta_i \delta \dot{q}_i)$

$$\delta R = \sum_{i=k+1}^{n} \frac{\partial L}{\partial q_i} \delta q_i + \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i + \frac{\partial L}{\partial t} \delta t - \sum_{i=1}^{K} \left(\frac{\partial L}{\partial q_i}\right) \delta \dot{q}_i - \sum_{i=1}^{K} \dot{q}_i \delta \beta_i$$

$\therefore \delta R = \sum_{i=k+1}^{n} \frac{\partial L}{\partial q_i} \delta q_i + \sum_{i=k+1}^{n} \frac{\partial L}{\partial \tilde{q}_i} \delta \dot{q}_i + \sum_{i=1}^{k} (-\dot{q}_i) \delta \beta_i + \frac{\partial L}{\partial t} \delta \mathrm{t}$ .... (4)

Compute the like co-efficient of (3) & (4) we get,

$$\frac{\partial L}{\partial q_i} = \frac{\partial R}{\partial q_i}$$
$$\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial R}{\partial \dot{q}_i}$$
$$\dot{q}_i = -\frac{\partial R}{\partial \beta_i}$$
$$\frac{\partial L}{\partial t} = \frac{\partial R}{\partial t}$$

∴ By Lagrangian equation of motion,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, \text{ for } i = k+1, k+2, \dots n$$
$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_i}\right) - \frac{\partial R}{\partial q_i} = 0, \text{ for } i = k+1, k+2, \dots n$$

This is known as Lagrangian Routhian equation of motion for Function.

Application: 1

(1) There are $n-k,$ second order diff equation in the non-integral variable.

Thus the Routhian procedure succeeded in eliminating the ignorable ordinate from the equation.

(2) If it's the effect the no of degree freedom has been reduced the $n-k$. It held to obtain expression for k ignorable co-ordinate.

$\dot{q}_i = \frac{\partial R}{\partial B_i}$, Integrate $q_i = -\int \frac{\partial R}{\partial \beta_i} dt$, for $i = 1,2, \dots \dots, k$

Example 1:

Obtain the equation of the motion using Routhian function for the Kepler's problem.

Solution:

Write up to this as in the Kepler's problem

Here, $L = \frac{1}{2}\left[\dot{r}^2 + r^2\dot{\theta}^2\right] + \frac{M}{r}$.

$\frac{\partial L}{\partial \theta} = 0 \Rightarrow \theta$ is ignorable coordinate. By Lagrangian eq of motion.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - 0 = 0.$$

Integrate, $\frac{\partial L}{\partial \dot{\theta}} = \beta$

$$\Rightarrow r^2\dot{\theta} = \beta$$
$$\Rightarrow \dot{\theta} = \frac{\beta}{r^2}$$

Eliminate θ between (1) & (2)

$$L = \frac{1}{2}\left[\dot{\gamma}^2 + \gamma^2 \cdot \frac{\beta^2}{r^4}\right] + \frac{\mu}{r}.$$
$$L = \frac{1}{2}\left[\dot{r}^2 + \frac{\beta^2}{\gamma^2}\right] + \frac{\mu}{r}.$$

By definition of Routhian form $R = \mathrm{L} - \sum_{i=1}^{k} \beta_i \dot{q}_i$

$$R = \frac{1}{2}\left[\dot{r}^2 + \frac{\beta^2}{r^2}\right] + \frac{\mu}{\gamma} - \beta(\dot{\theta})$$
$$R = \frac{1}{2}\left[\dot{\gamma}^2 + \frac{\beta^2}{\gamma^2}\right] + \frac{\mu}{r} - \beta \cdot \frac{\beta}{r^2}$$

$$R = \frac{1}{2}\left[\dot{r}^2 - \frac{\beta^2}{r^2}\right] + \frac{\mu}{r}$$

$\therefore$ By Lagrangian eqn of motion for Routhian function is $\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{r}}\right) - \frac{\partial R}{\partial r} = 0$. (non-ignorable function) $\frac{d}{dt}(\dot{\gamma}) - \left[\frac{-1}{2}\beta^2 \cdot \frac{(-2)}{\gamma^3} - \frac{\mu}{\gamma^2}\right] = 0$

$$\ddot{\gamma} - \frac{\beta^2}{\gamma^3} + \frac{\mu}{\gamma^2} = 0$$

Which is Same as Lagrangian equation n of motion.

Example 2:

Obtain Jacobi Integral (or) Obtain energy integral for a conservative system.

Proof:

Consider a standard non-holonomic form of Lagrangian equation

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \sum_{j=1}^{m} \lambda_j a_{ji}$ (for $i = 1, \cdots n$) ………… (1)

where $L = L(q_i, \dot{q}_i)$ is not explicit function of time t.

Let us write m equation of constraints in the form

$$\sum_{j=1}^{m} a_{ji}\dot{q}_i = 0, j = 1,2, \dots \dots m$$

(i.e.,) $a_{ji} = a_{ji}(q_i, t)$

Any holonomic constraint from $\phi_j(q)$ can't be explicit function of time t.

$\therefore a_{ij} = \frac{\partial \phi j}{\partial t} = 0$ …………..(*)

Now, $L = L(q_i, \dot{q}_i)$

Take total diff w.r.t t.

$\frac{dL}{dt} = \sum_{i=1}^{n} \frac{\partial L}{\partial q_i}\dot{q}_i + \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i$ ………… (2)

By (1) $\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \sum_{j=1}^{m} \lambda_j a_{ji}$ ………… (3)

Sub (3) in (2) $\frac{dL}{dt} = \left[\sum_{i=1}^{n} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \sum_{j=1}^{m} \lambda_j a_{ji}\right]\dot{q}_i + \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i$

Integrable,

$$L = \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i + h \text{ (constant of int)}$$

$$\text{(or) } h = L - \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i$$

This constant h is called Jacobi Integral (oR) Energy Integrable.

This can be written as $h = T_2 - T_0 + V$

State the condition of natural system.

A natural system is a conservative system. Which has additional property,

(1) describe the standard holonomic form of Lagrangian differentiation eqn.

(2) The K.E is expressed as a holonomic quartic function of $q_i{}'$ s.

$$T = T_2 = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{m} m_{ij} q_i q_j$$

Where $m_{ij} = m_{ij}(q)$ not a function of time.

∴ Jacobi Integral of this system is equal to total energy.

(i.e.,) $T + v = h$.

⇒ Total energy is conserved.

Definition: Liouville's System.

Let $T = \frac{1}{2} f \sum_{i=1}^{m} m_i(q_i) \dot{q}_i^2$

$$v = \frac{1}{f} \sum_{i=1}^{n} v_i(q_i)$$

where $m_i(q_i) > 0$ and $f = \sum_{i=1}^{n} f_i(q_i) > 0$.

This system having T and V above form is called Liouville's system.

Application of Integral of motion:

Reduce the spherical pendulum to quadrature and obtain Integral of motion.

Write up to this as in Spherical pendulum problem.

$$L = \frac{1}{2}ml^2[\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta] - mgh\cos\theta$$

We know that , the given system is conservative holonomic with n-degree of freedom and

$(n-1)$ coordinate is ignorable.

Then the system is completely integrable.

Here the given system all the condition of conservative system.

Here we have a conservative holonomic system having two degrees of freedom and one ignorable co-ordinate. Hence it can be follow solved completely by quadrature's.

$$\frac{\partial L}{\partial \dot{\phi}} = ml^2\dot{\phi}\sin^2\theta = \alpha_\phi \text{ .... (1)}$$

where α_ϕ is a constant of given tum to ϕ general momentum correspond to ϕ

By Routhian function,

$$R = L - \sum_{i=1}^{K} \beta_i \dot{q}_i$$

$$R = L - \alpha_\phi \dot{\phi}$$

Eliminate $\dot{\phi}$ from (1) & (2)

$$R = \frac{1}{2}ml^2\left[\dot{\theta}^2 + \frac{\alpha_\phi^2}{m^2 l^4 \sin^4\theta}\sin^2\theta\right] - mgl\cos\theta - \frac{\alpha_\phi\,\alpha_\phi}{\text{m}\,l^2\sin^2\theta}$$

$$R = \frac{1}{2}ml^2\dot{\theta}^2 - \frac{1}{2}\cdot\frac{\alpha_p^2}{ml^2\sin^2\theta} - mgl\cos\theta$$

$R = T' - V'.$

where $T' = \frac{1}{2} ml^2\dot{\theta}^2$

$$v' = \frac{1}{2} \cdot \frac{\alpha_\phi^2}{ml^2 \sin^2\theta} + mgl\cos\theta$$

The form of T' and V' is that of natural System having one degree of freedom

$$h = \tau' + v' = \frac{1}{2} ml^2\dot{\theta}^2 + \frac{1}{2}\frac{\alpha_\phi^2}{ml^2 \sin^2\theta} + mgl\cos\theta$$

$$\therefore \frac{1}{2} ml^2\dot{\theta}^2 = h - \frac{\alpha_\phi^2}{2ml^2 \sin^2\theta} - mgl\cos\theta.$$

$$\dot{\theta}^2 = \frac{2}{ml^2}\left[h - mgl\cos\theta - \frac{\alpha\phi^2}{2ml^2 \sin^2\theta}\right]$$

$$\dot{\theta}^2 = \frac{2}{ml^2 2\, ml^2 \sin^2\theta}[2ml^2 \sin^2\theta(h - mgl\cos\theta) - \alpha\phi^2]$$

$$\dot{\theta} = \frac{1}{ml^2 \sin\theta}\sqrt{(2ml^2 \sin^2\theta[h - mgl\cos\theta) - \alpha_\phi^2]}$$

Cross multiply and integrate.

$$\int_{\theta_0}^{\theta} \frac{ml^2 \sin\theta d\theta}{\sqrt{2ml^2 \sin^2\theta[h - mgl\cos\theta) - \alpha_\phi^2}} = \int_{t_0}^{t} dt = t - t_0$$

The motion in θ is liberation for $0 \le \theta \le \pi$ Hence the sign of the system root should same as that of $d\theta$

By (1) $ml^2 \sin\theta\, \dot{\phi} = \alpha_\phi$

$$\alpha_\phi = \frac{\alpha_\phi dt}{ml^2 \sin^2\theta}$$

(i.e.,) $ml^2 \sin\theta = \dfrac{\alpha_\phi dt}{\sin\theta d\phi}$

$$\frac{\sqrt{2ml^2\sin^2\theta(h - mgl\cos\theta) - \alpha_\phi^2}}{\dot\theta} d\phi = \frac{\alpha_\phi dt}{\sin\theta}$$

(i.e.,) $\dfrac{\alpha_\phi d\theta}{\sin\sqrt{2ml^2\sin^2\theta(h - mgl\cos\theta) - \alpha_\phi^2}} = \int_{\phi_0}^{\phi} d\phi = \phi - \phi_0$

where $\phi_0 = \phi(t_0)$

Example: 3

Let us assume that system which as

$$K.E \Rightarrow T = \frac{1}{2} f \sum_{i=1}^{n} \dot q_i^2 = \frac{f}{2}[\dot q_1^2 + \dot q_2^2 + \cdots \ldots . + \dot q_n^2]$$

$$P.E \Rightarrow V = \frac{1}{f} \sum_{i=1}^{n} v_i(q_i) = \frac{1}{f}[v_1(q_1) + v_2(q_2) + \cdots + v_n(q_n)]$$

When f= $\sum_{i=1}^{n} f_i(q_i) > 0 \qquad f_1(q_1) + \cdots + f_n(q_n).$

Prove that the above System is Separable.

Proof:

By Lagrangian differentiation equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0$ ....(1)

$(\because L = T - V)$

we obtain, $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot q_i}\right) - \dfrac{\partial L}{\partial q_i}$ =0

$\frac{d}{dt}(f \dot q_i) - \frac{1}{2}\left(\frac{\partial f_i}{\partial q_i}\right) \sum_{i=1}^{n} \dot q_i^2 + \frac{1}{f}\frac{\partial v_i}{\partial q_i} - \frac{v}{f^2}\frac{\partial f_i}{\partial q_i} = 0$ .... (2)

Sub T, v in (2).

$\Rightarrow$ This is a natural system so if has an energy interal equation is given $T + v = h$

$\frac{f}{2}\sum_{i=1}^{n} \dot{q}_i^2 + v = h$ .... (3)

$\frac{1}{2}\sum_{i=1}^{n} \dot{q}_i^2 = \frac{1}{f}(h - v)$ .... (4)

Sub (4) in (2).

$$\frac{d}{dt}(fq_i) - \frac{\partial f_i}{\partial q_i}\cdot\frac{1}{f}(h-v) + \frac{1}{f}\frac{\partial v_i}{\partial q_i} - \frac{v}{f^2}\frac{\partial f_i}{\partial q_i} = 0$$
$$\frac{d}{dt}(f\dot{q}_i) - \frac{h}{f}\cdot\frac{\partial f_i}{\partial q_i} + \frac{v}{f}\frac{\partial f_i}{\partial q_i} + \frac{1}{f}\frac{\partial v_i}{\partial q_i} - \frac{v}{f^2}\frac{\partial f_i}{\partial q_i} == 0$$

$$(x2f\dot{q}_i) \Rightarrow 2f\dot{q}_i\frac{d}{dt}(f\dot{q}_i) - 2f\dot{q}_i\frac{b}{f}\frac{\partial f_i}{\partial q_i} + 2f\dot{q}_i + \frac{1}{f}\frac{\partial v}{\partial q_i} = 0$$

$$\frac{d}{dt}(f\dot{q}_l)^2 - 2h\frac{\partial f_i}{\partial q_i}\dot{q}_i + 2\frac{\partial v_i}{\partial q_i}\dot{q}_i = 0.$$
$$\frac{d}{dt}(f\dot{q}_i)^2 - 2\frac{d}{dt}(hf_i) + 2\frac{d}{dt}(v_i) = 0$$
(ie) $\frac{d}{dt}(f^2\dot{q}_i^2)^2 - 2\frac{d}{dt}(hf_i - v_i) = 0.$

Integrate,

$$f^2\dot{q}_i^2 - 2[hf_i(q_i) - v_i(q_i) + c_i]_{i=1} = 0$$

we have (Take $\sum$)

$$\frac{1}{2}\sum_{i=1}^{n} f^2\dot{q}_i^2 = h\sum_{i=1}^{n} f_i(q_i) - \sum_{i=1}^{n} v_i(q_i) + \sum_{i=1}^{n} c_i$$
$$\div f^2 \Rightarrow \frac{1}{2}\sum_{i=1}^{n} q_i^2 = \frac{hf}{f^2} - \frac{v}{f^2} + \frac{1}{f^2}\sum_{i=1}^{n} c_i$$
$$\frac{h}{f} - \frac{v}{f} = \frac{h}{f} - \frac{v}{f} + \frac{1}{f^2}\sum_{i=1}^{n} c_i \quad \text{(by (4))}$$
$$\Rightarrow \sum_{i=1}^{n} c_i = 0.$$

Hence C 's and h together comprise n independent constant of motion, the remaining n intergral equation of motion are obtained by using eqn (5) in the form,

$$\frac{dq_i}{dt} = \frac{\sqrt{2(hf_i - v_i + c_i)}}{f}$$

$$\frac{dq_1}{\sqrt{2(hf_1 - v_1 + c_1)}} = \frac{dq_2}{\sqrt{2(hf_2 - v_2 + c_2)}} = \cdots = \frac{dq_n}{\sqrt{2(hf_n - v_n r'}} = \frac{dt}{f}$$

$$\Rightarrow \frac{dq_1}{\sqrt{2(hf_1 - v_1 + c_1)}} = \frac{dq_n}{\sqrt{2(hf_2 - v_2 + c_2)}} = \frac{dt}{f} = d\tau,$$

∴ The given system is separable.

Example: 4 [Mass Spring System]

Suppose a mass spring system is attached to a frame which is translating with uniform velocity v_0. (as in figure). Let l_0 be the unstressed spring length and use the elongation x as the generalized coordinate. Find the Jacobi integral for this system.

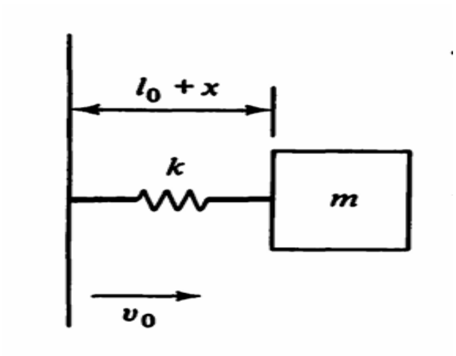


Figure.2.6. A translating mass spring System

Solution:

The kinetic energy is,

$$T = \frac{1}{2}m(v_0 + \dot{x})^2$$

$$T = \frac{1}{2}m\dot{x}^2 + m\dot{x}v_0 + \frac{1}{2}mv_0^2$$

$$= T_2 + T_1 + T_0$$

Where, $T_2 = \frac{1}{2}m\dot{x}^2$ since $v = \dot{l}_0 + \dot{x}$, $v = (v_0 + \dot{x})$

$$T_1 = mv_0\dot{x}$$

$$T_0 = \frac{1}{2}mv_0^2$$

The potential energy,

$$V = \frac{1}{2}kx^2 \text{ (By Hooke's Law)}$$

The mass-spring system meets all the conditions of a holonomic conservative system.

Since T and v are not explicit functions of time

$\therefore Q_x$ is only generalized co-ordinate derived from v.

Although the moving frame does work on system resulting in a changing total energy

$T + v$. $\therefore$ Jacobi integral exists and is equal to,

$$T_2 - T_0 + v = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}mv_0^2 + \frac{1}{2}kx^2 = h$$

where h is constant, T_0 is constant.

$\therefore T_2 + v$ is also constant.

Example 5:

A small tube, bent in the form of a circle of radius r, rotates about a vertical diameter with a constant angular velocity w. A particle of masses m can slide without Friction inside the tube. At any given time, the configuration of the system is specified by the angle θ which is measured from the upward vertical to the line connecting the center 0 and the particle. Find the Jacobi Integral.

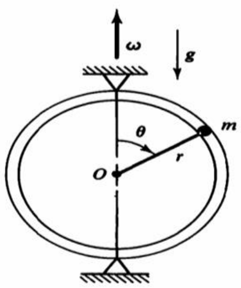


Figure.2.7.

Solution:

Let us assume a fixed Cartesian frame with its origin at O and z axis verticle.

The plane of the tube coincides with the xz-plane at $t = 0$.

The transformation equation relating generalized co-ordinate θ and the position of the particle $p(x, y, z)$ are given by,

$$x = r\sin\theta\cos\omega t$$
$$y = r\sin\theta\sin\omega t$$
$$z = r\cos\theta$$

This system is Rheonomic. It is also holonomic and has the same. number of degree of freedom are generalized co-ordinate. (i.e.) One (only one degree of freedom)

kinetic energy,
$$T = \frac{1}{2}mv^2$$
$$= \frac{1}{2}m(r^2\dot{\theta}^2 + r^2\omega^2\sin^2\theta)$$
$$T = T_2 + T_0$$

where, $T_2 = \frac{1}{2}mr^2\dot{\theta}^2$

$$T_0 = \frac{1}{2}mr^2\omega^2\sin^2\theta$$

Potential Energy, $V = mgr\cos\theta$.

∴ The Lagrange's function is,

$$L = T - V$$
$$L = \frac{1}{2}m(r^2\dot{\theta}^2 + r^2\omega^2\sin^2\theta) - mgr\cos\theta.$$

Here L is not an explicit function of t even though the system Rheonomic.

Hence the system is conservative. ∴ It's Jacobi integral is given by,

$$T_2 - T_0 + v = h$$
$$\text{(i.e.) } \frac{1}{2}mr^2\dot{\theta}^2 - \frac{1}{2}mr^2\omega^2\sin^2\theta + mgr\cos\theta = h$$

Example 6:

Two particles each of mass ' m ' are connected by a rigid massless rod of length ' l ' as in figure. The particles are supported by a knife-edge placed perpendicular to the rod. Assuming that all motion is confined to the horizontal xy-plane, Find the Jacobi integral.

Solution:

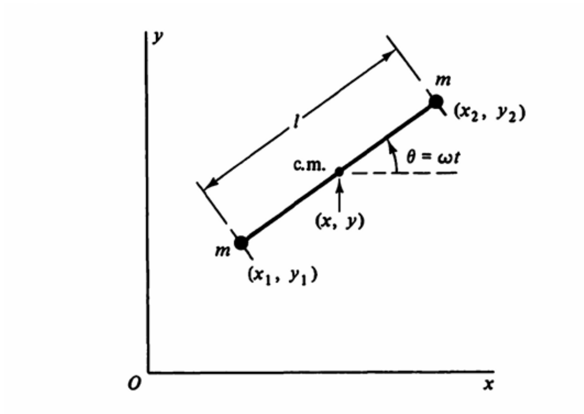


Figure.2.8. A non-holonomic Rheonomic system

Let (x, y) be the generalized co-ordin of the system.

$\therefore$ Potential Energy is given by $v = 0$.

$\therefore$ But kinetic energy is given by, $T = m(\dot{x}^2 + \dot{y}^2) + \frac{1}{4}ml^2\omega^2$

But the non-holonomic constraint is, $\cos \omega t dx + \sin \omega t dy = 0$

$\therefore$ Jacobi integral,

$$h = T_2 - T_0 + v$$

$$h = m(\dot{x}^2 + \dot{y}^2) - \frac{1}{4}ml^2w^2$$

Example 7:

An inverted pendulum consists of a particle of mass m supported by a rigid mass less rod of length l. The piv of 0 has a vertical motion given by $z = A\sin \omega t$. of tain the Lagrangian function and find the differential equation of motion.

Solution:

Kinetic energy of the rod.

$$T = \frac{1}{2}m(l\dot{\theta})^2$$
$$= \frac{1}{2}ml^2\dot{\theta}^2$$

Since the pivot 0 has a

$z \uparrow (\because z =$ Asimov

$$z = A\sin\omega t$$
$$\dot{z} = -A\omega\cos\omega t$$
$$\ddot{z} = -A\omega^2\sin\omega t$$

Net acceleration is $(g - A\omega^2\sin\omega t)$

The potential energy of the system is

$$V = ml\cos\theta(g - A\omega^2\sin\omega t)$$
$$L = T - V$$
$$L = \frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta + m\ell\cos\theta A\omega^2$$

The equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0.$$
$$\frac{d}{dt}(ml^2\dot{\theta}) - mgl\sin\theta + ml\sin\theta A\omega^2\sin\omega t =$$
$$\therefore ml^2\ddot{\theta} + mlA\omega^2\sin\theta\sin\omega t - mgl\sin\theta = 0$$

UNTT-III

Hamilton's Equations: Hamilton's Principle - Hamilton's Equations - Other variation principles.

Chapter 3: Sections 3.1 to 3.3

3.1. Hamilton's Principle:

The mathematics of extremum problems is partially covered in ordinary calculus and partially in the calculus of variations. Before we enter into a discussion of Hamilton's principle, let us review briefly some of the mathematical concepts associated with these problems.

Stationary value of function:

Let $f = f(q_1 \dots - q_n)$ be a continues function having second order partial derivatives.

$\therefore$ First variation of f at q_0 is $\delta f = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i}\right)_0^{\delta q} = 0$

[Where zero subscript indicate at reference pt]

$\therefore$ The necessary and sufficient condition that f have stationary value at q_0 is that $\delta f = 0$ for all points $\delta q's$.

where $q = q_0 + \delta q$

If $\delta q's$ are independent and reversible then$\left(\frac{\partial f}{\partial q_i}\right)_0 = 0,\ i = 1,2 \dots n.$ is called a stationary point.

Find the method of Finding Stationary Value:

Method (1): Lagrange Multiplier methods

Consider free variation of an augmented function $F = f(q_1, q_2 \dots q_n, \lambda_1 \dots \lambda_m)$

$$F = f + \sum_{j=1}^{m} \lambda_i \phi_j$$

where n q's and $m\ \lambda's$ are regarded as independent variable.

$\therefore$ The necessary & sufficient condition for f to be stationary is$\left(\frac{\partial F}{\partial q_i}\right)_0 = 0, i = 1,2 \dots n$ and

$\left(\frac{\partial F}{\partial \lambda_j}\right)_0 = 0, \quad j = 1,2 \cdots m.$

Example: 1

Find the stationary value of function $f = z$ subject to the constraint

$\phi_1 = x^2 + y^2 + z^2 - 4 = 0\ , \phi_2 = xy - 1 = 0$

Solution:

Given, $f = z$

Let $\phi_1 = x^2 + y^2 + z^2 - 4 = 0$

$\phi_2 = xy - 1 = 0$

Let $F = f + \lambda_1\phi_1 + \lambda_2\phi_2$

$F = z + \lambda_1(x^2 + y^2 + z^2 - 4) + \lambda_2(xy - 1)$

For stationary values $\left(\frac{\partial F}{\partial q_i}\right)_0 = 0$ $\left(\frac{\partial F}{\partial \lambda_j}\right)_0 = 0$.

$\frac{\partial F}{\partial x} = 0 \Rightarrow 2x\lambda_1 + y\lambda_2 = 0$ .... (1)

$\frac{\partial F}{\partial y} = 0 \Rightarrow 2y\lambda_1 + \mathrm{x}\lambda_2 = 0$ .... (2)

$\frac{\partial F}{\partial z} = 0 \Rightarrow 1 + 2z\lambda_1 = 0$ .... (3)

$\frac{\partial F}{\partial \lambda_1} = 0 \Rightarrow x^2 + y^2 + z^2 - 4 = 0$ .... (4)

$\frac{\partial F}{\partial \lambda_2} = 0 \Rightarrow xy - 1 = 0$ .... (5)

$(1)\,x - (2)y \Rightarrow 2\lambda_1(x^2 - y^2) = 0.$

$\Rightarrow x^2 - y^2 = 0$

$\Rightarrow x = y$

Sub $\phi_1 \Rightarrow 2x^2 + z^2 = 4$

$z^2 = 4 - 2x^2$

Where $x = 1, z^2 = 2, z = \pm\sqrt{2}$

$x = -1, z^2 = 2, z = \pm\sqrt{2}$

where $x = 1, y = 1, z = \pm\sqrt{2}$

$x = -1, y = -1, z = t\sqrt{2}$

The required Stationary points are

$(1, 1, +\sqrt{2}), (-1, -1, +\sqrt{2})$

$(1,1, -\sqrt{2}), (-1, -1, -\sqrt{2}).$

From (2) $\frac{2y}{x} = \frac{-\lambda_2}{\lambda_1}$

From (1) $\frac{2x}{y} = \frac{-\lambda_2}{\lambda_1}$

$$\Rightarrow \frac{2y}{x} = \frac{2x}{y}$$

$\Rightarrow x^2 = y^2 \Rightarrow x = y$

Where $x = 1, y = 1, \frac{2}{1} = \frac{-\lambda_2}{\lambda_1}$

$\lambda_2 = -2\lambda_1$

By equation (3) $1 + 2z\lambda_1 = 0$

$\Rightarrow 2z\lambda_1 = -1$

$\Rightarrow \lambda_1 = \frac{-1}{2z}$ ………….. (6)

$\lambda_2 = -2\left(\frac{-1}{2z}\right)$

$\lambda_2 = \frac{1}{z}$ …………… (7)

(∵ from equation (6) and (7))

Hence $\lambda_1 = \frac{\pm 1}{2\sqrt{2}}$, $\lambda_2 = \frac{\pm 1}{\sqrt{2}}$ are the required Lagrange Multipliers.

At all points $(1,1,\pm\sqrt{2}), (-1,-1,\pm\sqrt{2})$ f attains maximum value.

Theorem 1:

Derive-Euler-Lagrange Equation (or)

Derive the necessary and sufficient condition for a stationary value of a definite integral.

Proof:

Suppose that, we wish to find the necessary condition for a stationary value of a definite integral

$I = \int_{x_0}^{x_1} F[y(x), y'(x), x]dx$

Where, $y'(x) = \frac{dy}{dx}, x_0$ and x_1 are fixed

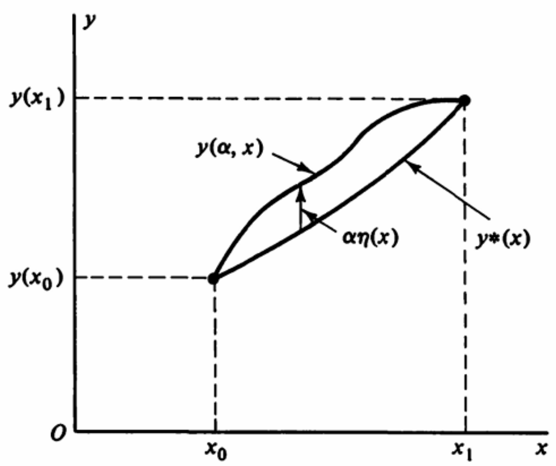


Fig-3.1 The variation of a curve between fixed end-points

here $x_0 = x(x_0), y_0 = y(y_0)$

$x_1 = x(x_1), y_1 = y(y_1)$

To find a function $y^*(x)$ which gives a stationary value for I.

We have

$y(x) = y^*(x) + \delta y(x)$

where $\delta y(x)$ is a small variation in $y(x)$

Put $\delta y(x) = \alpha\eta(x)$

For any given $\eta(x)$

We can consider varied curve y to be a function of α and x

$y(\text{x}, \alpha) = y^*(x) + \alpha\eta(x)$

Suppose that $\delta y = 0$ at the and points

(i.e.,) $\eta(x_0) = \eta(x_1) = 0$

$\Rightarrow y(x_0)$ and $y(x_1)$ ane fixed.

For any given $\eta(x)$. I is a function of α only.

A necessary condition that $y^*(x)$ result is a stationary value of I, is that its first variation is Zero.

(i.e.,), $\delta_I = \left(\frac{dI}{d\alpha}\right)_{\alpha=0}$

for arbitrary $\eta(x), \alpha \neq 0$.

$$\frac{dI}{dx} = \frac{d}{dx}\int_{x_0}^{x_1} f(y, y', x)dx$$

$$\frac{dI}{d\alpha} = \int_{x_0}^{x_1} \left(\frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'} \cdot \frac{\partial y'}{\partial \alpha}\right) dx$$

$$\frac{dI}{d\alpha} = \int_{x_0}^{x_1} \left(\frac{\partial f}{\partial y}\, \eta(x) + \frac{\partial f}{\partial y'}\, \eta'(x)\right) dx \quad \ldots\ldots\ldots\ldots.. (1)$$

But, $\int_{x_0}^{x_1} \frac{\partial f}{\partial y'} \eta'(x)dx = \left[\frac{\partial f}{\partial y'}\eta(x)\right]_{x_0}^{x_1} - \int_{x_0}^{x_1} \eta(x)\, \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) dx$

$$= 0 - \int_{x_0} \eta(x) \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) dx$$

$\therefore \eta(x_1) = \eta(x_0) = 0$

Sub in (1)

$$\Rightarrow \frac{dI}{dx} = \int_{x_0}^{x_1} \left[\frac{\partial f}{\partial y}\eta(x) - \eta(x)\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right] dx.$$

$$= \int_{x_0}^{x_1} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right] \eta(x)dx$$

But the stationary value of I_1,

$$\delta I = 0$$

$$\Rightarrow \frac{dI}{d\alpha} = 0$$

$$\Rightarrow \int_{x}^{x_0} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right] \eta(x) dx = 0$$

$$\Rightarrow \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right] \eta(x) = 0. [\because \eta(x) \neq 0]$$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

For any curve $y = y^*(x)$.

This is the necessary condition for I to be Stationary.

This is also known as, Euler-Lagrange's Equation.

Theorem 2:

Derive Euler Lagrange differential equation for stationary value of a definite integral in. many variables.

Proof:

Suppose that we wish to find the necessary Condition for the stationary value of

$$\mathrm{I} = \int_{x_0}^{x_1} f(\dot{y}_1, y_2, \cdots - y_n, y_1', y_2' - \cdots - y_n', x) dx$$

$where\ y_i' = \frac{\partial y_i}{\partial x_i}$, x_0, x_1 are fixed.

proceed as in the above, by replacing by,

$y_i(x), y_i'(x), \eta_i(x)$

Explain the term Brachistochrone:

A Brachistochrone is a curve joining two points along which a particle moves under the action of a given conservative force field in the least positive time.

Note:

Brachisto - Shortest

Chrone - time.

Theorem 3:

State and prove Brachistochrone problem:

The Brachistochrone problem is one of the Classical problem of the calculus of variation. (or) The problem is to find a curve $y(x)$ between the origin ' O ' and the point (x_1, y_1) such that a particle starting from rest at O d and sliding down the curve without friction under the influence of a uniform gravitational field, will reach the end of the curve in a minimum time

Proof:

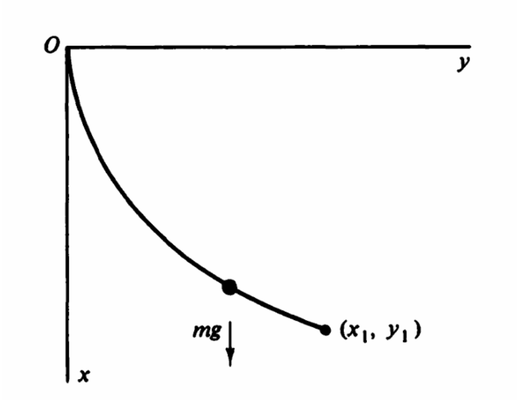


Fig.3.2. The Brachistochrone problem

Let us assume that the gravitational force is directed along the positive x-axis.

To find the velocity v as a function of position By using "principle of conservation of Energy".

"Change in kinetic energy = work done".

(i.e.,) $\frac{1}{2}\mathrm{m}v^2 = mgx$

(i.e.,) $v^2 = 2gx$

$v = \sqrt{2gx}$

Arc length $\Rightarrow ds = \sqrt{1+y'^2}dx$

put $v = \dfrac{ds}{dt}$

$dt = \dfrac{ds}{v}$

$\mathrm{t} = \int_0^x \dfrac{\sqrt{1+y'^2}}{\sqrt{2gx}}$ where $dt = \dfrac{\sqrt{1+y'^2}}{\sqrt{2gx}}dx$

Let $f(y, \mathrm{y}', x) = \frac{\sqrt{1+y'^2}}{\sqrt{2gx}}$ Since f is not an expliat fund ion of $y, \frac{\partial f}{\partial y} = 0$.

$$\frac{\partial f}{\partial y'} = \frac{1}{\sqrt{2gx}} \cdot \frac{1}{2\sqrt{1+y'^2}} \cdot 2\mathrm{y}_1$$

$$\frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{2g\mathrm{x}(1+y'^2)}}$$

By "Euler's Lagrange’s differentiation equation",

$$\frac{d}{dt}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0 \ ,$$

(i.e.,) $\frac{d}{dt}\left(\frac{\partial f}{\partial y'}\right) = 0 \left[\because \frac{\partial f}{\partial y} = 0\right]$

(i.e.,) Integrate,

$$\frac{\partial f}{\partial y'} = \text{ constant } = \text{c}$$

(i.e.,) $\frac{y'}{\sqrt{2gx(1+y_1^2)}} = c$

Square and cross multiply,

$$y'^2 = 2gxc^2(1+y_0^2) \Rightarrow y'^2 = 2gxc^2 + 2g'y'^2c^2$$

$$y'^2(1-2gxc^2) = 2gxc^2 \Rightarrow 2gxc^2 = y'^2(1-2gxc^2$$

(i.e.,) $y'^2 = \frac{2gxc^2}{1-2gxc^2}$

(i.e.,) $y' = \frac{\sqrt{2gxc^2}}{\sqrt{1-2gxc^2}}$

(i.e.,) $\frac{dy}{dx} = \frac{\sqrt{2gxc^2}}{\sqrt{1-2gxc^2}}$

put $x = a(1-\cos\theta), a = \frac{1}{4gc^2} \ dx = a\sin\theta d\theta.$

$$\frac{dy}{a\sin\theta d\theta} = \frac{\sqrt{2gc^2 \cdot \frac{1}{4gc^2}(1-\cos\theta)}}{\sqrt{1-2gc^2 \cdot \frac{1}{4gc^2}(1-\cos\theta)}}$$

$$= \frac{\sqrt{\frac{1}{2}(1-\cos\theta)}}{\sqrt{1-\frac{1}{2}(1-\cos\theta)}}$$

$$\frac{dy}{a\sin\theta d\theta} = \frac{\sqrt{\frac{1}{2}(1-\cos\theta)}}{\sqrt{\frac{1}{2}(1+\cos\theta)}} = \frac{\sqrt{(1-\cos\theta)(1-\cos\theta)}}{\sqrt{(1+\cos\theta)(1-\cos\theta)}}$$

$$\frac{dy}{a\sin\theta d\theta} = \frac{1-\cos\theta}{\sqrt{1-\cos^2\theta}}$$

$$\frac{dy}{a\sin\theta d\theta} = \frac{1-\cos\theta}{\sin\theta}$$

(i.e.,) $dy = a(1-\cos\theta)d\theta.$

Integrate, $y = a(\theta - \sin\theta)$

This is the required path.

hence $x = a(1 - \cos\theta)$

$y = a(\theta - \sin\theta)$ which is cycloid.

A cycloid path lead to a stationary value of t.

This path is minimum time

Theorem 4: Tautachrone problem (Geodesic problem)

Prove that Geodenic sphere are great circle (or) Find the shortest path between two points in a given space.

Let us consider the problem of finding the path of minimum length c shortest distance between two given points on the two dimension surface of a sphere of radius r.

Proof:

Let us use the spherical coordinate (θ, ϕ) as variables.

∴ The spherical transformation is,

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

$$ds^2 = r^2 d\theta^2 + r^2\sin^2\theta d\phi^2$$

$$ds^2 = r^2 d\theta^2\left[1 + \sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2\right]$$

$$ds^2 = r^2 d\theta^2[1 + \sin^2\theta\phi'^2] \text{ where } \phi' = \frac{d\phi}{d\theta}.$$

Integrate,

$$S = r\int_{\theta_1}^{\theta_2}\sqrt{1 + \phi'^2\sin^2\theta d\theta}$$

Let $f = \sqrt{1 + \phi'^2\sin^2\theta}$

$$\frac{\partial f}{\partial\phi} = 0$$

$$\frac{\partial f}{\partial\phi'} = \frac{2\phi'\sin^2\theta}{2\sqrt{1 + \phi'^2\sin^2\theta}}$$

∴ Euler -Lagrangian equation of motion,

$$\frac{d}{dt}\left(\frac{\partial f}{\partial\phi'}\right) - \frac{\partial f}{\partial\phi} = 0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial f}{\partial\phi'}\right) = 0$$

Integrate, $\frac{\partial f}{\partial \phi'} = c.$

But, $C = \frac{\phi' \sin^2 \theta}{\sqrt{1+\phi_1^2 \sin^2 \theta}}$

Squaring on both sides,

$$c^2(1 + \phi'^2 \sin^2 \theta) = \phi^2 \sin^4 \theta$$
$$\phi'^2 \sin^2 \theta(\sin^2 \theta - c^2) = c^2$$
$$\phi'^2 = \frac{c^2}{\sin^2 \theta(\sin^2 \theta - c^2)}$$
$$\phi' = \frac{c}{\sin \theta \sqrt{\sin^2 \theta - c^2}}$$
$$d\phi = \frac{cd\theta}{\sin \theta \sqrt{\sin^2 \theta - c^2}}$$

$$d\phi = \frac{c \cos^2 \theta d\theta}{\sqrt{1 - c^2 \text{cosec}^2 \theta}}$$
$put\ x = c \cot \theta$
$$dx = -c\text{cosec}^2 \theta d\theta$$
Sub in above,

$$d\phi = \frac{-dx}{\sqrt{1 - c^2 - x^2}}$$

Integrate, $\phi = \cos^{-1}\left(\frac{x}{\sqrt{1-c^2}}\right) + \phi_0$

$$\therefore \phi - \phi_0 = \cos^{-1}\left(\frac{x}{\sqrt{1 - x^2}}\right)$$
$$\cos(\phi - \phi_0) = \frac{c \cot \theta}{\sqrt{1 - c^2}}$$

$$\cos \phi \cdot \cos \phi_0 + \sin \phi \cdot \sin \phi_0 = \frac{c\cot \theta}{\sqrt{1 - c^2}}$$
$$\Rightarrow r \sin \theta \cos \phi \cos \phi_0 + r\sin \theta \sin \phi \sin \phi_0 = \frac{cr\cos \theta}{\sqrt{1 - c^2}}$$
$$x = r\sin \theta \cos \phi$$
$$y = r\sin \theta \sin \phi$$
$$z = r\cos \theta$$
$$\Rightarrow x\cos \phi_0 + y\sin \phi_0 - \frac{cz}{\sqrt{1 - c^2}} = 0$$

This is of the form,

$Ax + By + Cz = 0$. (Plane equation)

Which is the equation of a plane through origin.

∴ The curve is great circle.

∴ The shortest path is a straight line.

Theorem 5:

State and prove Hamilton's principle

The actual path in configuration space followed by a holonomic dynamical system during the fixed interval t_0 to t_1. Such that the integral $I = \int_{t_0}^{t_1} L\,dt$ is stationary with respect to path variation which vanishes at the end points.

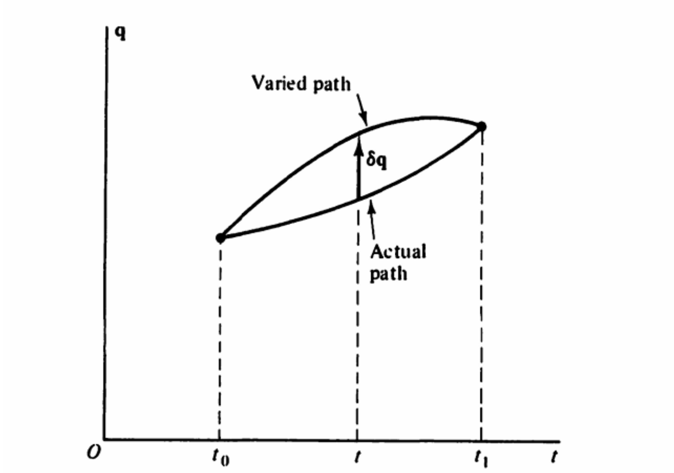


Fig-3.3 The actual and varied paths in extended configuration spaces

Consider a system of n particles with position $\vec{r}_1, \vec{r}_2, \ldots \vec{r}_n$ with respect to an inertial frame's.

By Lagrange from of 'D 'Alembert's principle.

$\sum_{i=1}^{n} \left(\vec{F}_i - m_i \ddot{\vec{r}}_i\right) \cdot \delta\vec{r}_i = 0$ …………. (1)

Where $\vec{F}_i$ is the applied force on ith particle

But $K.E = T = \frac{1}{2}\sum_{i=1}^{N} m_i \dot{r}_i^2$

$$\delta T = \frac{1}{2}\sum_{i=1}^{N} m_i 2\dot{\vec{r}}\delta\vec{r}_i$$

$\delta T = \sum_{i=1}^{N} m_i \dot{\vec{r}} \delta\vec{r}_i$ …………. (2)

Consider.

$\frac{d}{dt}\left(\sum_{i=1}^{\infty} m_i \dot{\vec{r}}_i \delta r_i\right) = \sum_{i=1}^{\infty} m_i \ddot{r}_i \delta\vec{r}_i + \sum_{t\to i}^{n} m_i \dot{\vec{r}} \delta \dot{r}_i$ ………..(*)

(1) $\Rightarrow \sum_{i=1}^{n} \vec{F}_i \cdot \delta\vec{r}_i = \sum_{i=1}^{n} m_i \ddot{\vec{r}}_i \delta\vec{r}_i$ ……….. (3)

Adding (2) and 3

$$\delta T + \sum_{i=1}^{N} (\vec{F_i} \cdot \delta \vec{r_i}) = \sum_{i=1}^{N} m_i \ddot{r}_i \cdot \delta \vec{r_i} + \sum_{i=1}^{N} m_i \dot{\vec{r}}_i \cdot \delta \dot{r}_i$$

$$= \frac{d}{dt}\left(\sum_{i=1}^{N} m_i \dot{\vec{r}}_i \cdot \delta \vec{r_i} \right) \quad (from\ eqn\ (*))$$

$$\therefore \delta T + \delta \omega = \frac{d}{dt}\left(\sum_{i=1}^{N} m_i \dot{\vec{r}}_i \cdot \delta \vec{r_i} \right)$$

Integrate on both sides with respect to time from t_0 to t_1 .

$$\int_{t_0}^{t_1} (\delta T + \delta \omega) dt = \left[\sum_{i=1}^{N} m_i \dot{\vec{r}}_i \delta \vec{r_i} \right]_{t_0}^{t_1}$$

here $\delta \vec{r_i} = 0$ at t_0 and t_1

$$\therefore \int_{t_0}^{t_1^1} (\delta T + \delta \omega) dt = 0$$

T and w are expressed interms of the generalized co-ordinates $(q_1, q_2 \ldots q_N)$ and their derivatives.

Suppose that applied forces are derivatives from the potential energy $V(q,t)$

then, $\delta w = -\delta v$.

$$\therefore \int_{t_0}^{t_1} (\delta T - \delta v) dt = 0$$

(i.e.,) $\int_{t_0^t}^{t} \delta(\tau - v) dt = 0$.

(i.e.,) $\int_{t_0}^{t} \delta L dt = 0$ [$\because$ Lagrangian $L = T - v$].

Further, if the System is holonomic (q's are independent)

$\therefore$ Integration and variation can be interchanged.

$$\therefore \delta \int_{t_0}^{t_1} L dt = 0.$$

(i.e.,) $\delta I = 0$

(i.e.,) I is stationary.

Hence proved Hamilton's principle.

Note:

We have proved that $\delta \int_{t_0}^{t_1} L dt = 0$. (ie) $\delta I = 0$. where $I = \int_{x_0}^{x_1} f(y(x), y'(x), x) dx$.

where $L = L(q, \dot{q}, \text{t})$

$y \to q$

$\dot{y} \to \dot{q}$

$\dot{x} \to \text{t}$

& $f \to L$

∴ By Euler's Lagrange's equation.

$$\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{y}_i}\right) - \frac{\partial f}{\partial y} = 0.$$

(i.e.,) $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$

This is Lagrangian differentiation equation.

3.2. Hamilton's Equations:

A new function $\text{H}(q, \text{p}, t)$ known as the Hamiltonian function , and use it to generate a set of first-order equations which are particularly symmetrical in form. These are the canonical equations of Hamilton.

Derive Hamilton's (Canonical) equation.

Let us consider a holonomic system with independent generalized co-ordinate $q_1, q_2 \ldots - q_n$. and obeying,

Lagrangian equation in the form,

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, i = 1,2 \ldots n$ .... (1)

Generalized momentum conjugate to $q_i's$ is given by $P_i = \frac{\partial L}{\partial \dot{q}_i}$

(i.e.,) $\dot{P}_i = \frac{\partial L}{\partial q_i}, i = 1$ to n ....(2)

By definition of Hamilton function.

$H = H(q, p, t)$ as

$H = \sum_{i=1}^{n} p_i \dot{q}_i - L(q, \dot{q}, t)$ ....(3)

$\delta H = \sum_{i=1}^{n} \frac{\partial H}{\partial q_i} \delta q_i + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} \delta p_i + \frac{\partial H}{\partial t} \delta t$

$By\ (3) \Rightarrow \delta H = \sum_{i=1}^{n} p_i \delta(\dot{q}_i) + \sum_{i=1}^{n} \dot{q}_i \delta p_i - \sum_{i=1}^{n} \frac{\partial L}{\partial q_i} \delta q_i - \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i - \frac{\partial L}{\partial \text{t}} \delta t$ ... ... (4)

$\delta H = \sum_{i=1}^{n} p_i \delta \dot{q}_i + \sum_{i=1}^{n} \dot{q}_i \delta p_i - \sum_{i=1}^{n} \dot{p}_i \delta q_i - \sum_{i=1}^{n} p_i \delta \dot{q}_i - \frac{\partial L}{\partial t} \delta t$

$\delta \text{H} = -\sum_{i=1}^{n} \dot{P}_i \delta q_i + \sum_{i=1}^{n} \dot{q}_i \delta p_i - \frac{\partial L}{\partial t} \delta t$ ... ... (5)

Compare (4) & (5), co-efficient,

$$\frac{\partial H}{\partial q_i} = -\dot{p}_i = -\frac{\partial L}{\partial q_i}$$
$$\frac{\partial H}{\partial p_i} = \dot{q}_i$$
and $\dfrac{\partial H}{\partial t} = \dfrac{-\partial L}{\partial t}$
$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$
$$\dot{q}_i = \frac{\partial H_i}{\partial p_i}$$

The Hamilton canonical equation of motion.

Example 1:

Show that for the conservative system holonomic (or) non-holonomic. Then Hamilton function is constants.

Proof:

By. Hamilton function, $H = \sum_{i=1}^{n} P_i \dot{q}_i - \mathrm{L}$.

and $P_i = \sum_{j=1}^{n} m_{ij} \dot{q}_i + a_i$

where m's and $q_i's$ are function of (q, t)

$$\therefore H = \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} \dot{q}_i \dot{q}_j + \sum_{i=1}^{n} \dot{q}_i a_j - L$$
$$H = (2T_2 + T_1) - (T - L)$$
$$= (2T_2 + T_1) - (T_2 + T_1 + T_0 - V)$$
$$= 2T_2 + T_1 - T_2 - T_1 - T_0 + V$$
$$H = T_2 - T_0 + V$$

Case (i):

For a conservative Holonomic System,

$$T_0 = 0, T_2 = T$$
$$\therefore H = T + V$$
$H =$ Total energy.

For such a system.

$H = H(q, p, t)$

(i.e.,) $\dot{H} = \sum_{i=1}^{n} \frac{\partial H}{\partial q_i} \dot{q}_i + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} \dot{p}_i + \frac{\partial H}{\partial t}$

$$= \sum_{i=1}^{n} \left(\frac{\partial H}{\partial q_i} \cdot \frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial p_i} \frac{\partial H}{\partial q_i} \right) + \frac{\partial H}{\partial t}$$

(By Hamilton canonical equation $\dot{p} = -\frac{\partial H}{\partial q_i}$ and $\dot{q} = -\frac{\partial H}{\partial p_i}$

$$= \sum_{i=1}^{n} (0) + \frac{\partial H}{\partial t}$$

(i.e.,)$\frac{dH}{dt} = \frac{\partial H}{\partial t}$.

∴ Total time derivative of the Hamilton is equal to the partial derivative of Hamilton.

For the conservative system $\mathrm{H} = \mathrm{H}(\mathrm{q}, t)$ and if L doesnot contain time.

$$\therefore \frac{\partial H}{\partial t} = 0$$

(i.e.,) $\frac{dH}{dt} = 0$

(i.e.,) $H =$ constant

∴ The consonative holonomic System

$H = T + V = a$ constant.

(i.e.,) $H = h$ (constant)

Where ' h ' is a Jacobian integral.

Case (ii):

For the conservative system of non-holonomic with ' m ' constraints represented by, $\sum_{i=1}^{n} a_{ij}\dot{q}_i = 0, \quad j = 1,2, \dots m$ ....(1)

For this system,

$\dot{P}_i = \frac{\partial L}{\partial q_i} + \sum_{j=1}^{n} \lambda_j a_{ji}$ where λ_j is an Lagrangian multiplier

$$\dot{P}_i = -\frac{\partial H}{\partial q_i} + \sum_{j=1}^{n} \lambda_j a_{ji}$$

(i.e.) $H = H(q, \mathrm{p}, t)$

$$\therefore \dot{H} = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial q_i}\dot{P}_i + \frac{\partial H}{\partial p_i}\dot{P}_i\right) \qquad \left(\because \frac{\partial H}{\partial t} = 0\right)$$

$$\dot{H} = \sum_{i=1}^{n} \left[\frac{\partial H}{\partial \hat{q}_i} \dot{q}_i + \frac{\partial H}{\partial p_i}\left(-\frac{\partial H}{\partial q_i} + \sum_{j=1}^{n} \lambda_j a_{ji}\right)\right]$$

$$= \sum_{i=1}^{n} \left(\frac{\partial H}{\partial q_i} \cdot \frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i}\right) + \sum_{j=1}^{n} \sum_{j=1}^{n} \frac{\partial H}{\partial p_i} \lambda_j a_{ji}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial H}{\partial p_i} \lambda_j a_{ji}$$

$$= \sum_{j=1}^{n} \lambda_j \left(\sum_{i=1}^{n} a_{ji}, \dot{q}_i\right)$$

$$\dot{H} = \sum_{j=1}^{n} \lambda_j (0) \qquad [\because \text{by } (1)]$$

(i.e.) $\frac{dH}{dt} = 0$

(i.e.) $H = \text{constant}.$

(i.e.) $H = h(= T_2 - T_0 + v).$

where h is Jacobian integral.

[For the natural system.

$$T_2 = T$$
$$T_0 = T_1 = 0$$
$$\therefore H = T + V = h(\text{ constant})]$$

Example 2:

Find the motion of a simple pendulum by using Hamiltonian equation.

Solution:

let ' l ' be the length of the string in which one end is fixed at ' A ' and another end is attached a Body of mass ' m '.

Let ' θ ' be the angle made by AP at time x with vertical.

Then its describe a simple pendulum.

$$S = l\theta$$
$$\dot{S} = l\dot{\theta}$$
$$\therefore T = \frac{1}{2} m v^2$$
$$T = \frac{1}{2} m l^2 \cdot \dot{\theta}^2$$
$$V = mgAc$$
$$= mg(l - l\cos\theta)$$
$$V = mgl(1 - \cos\theta)$$

∴ By Lagrangian function,

$$L = T - V.$$
$$L = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - \cos\theta)$$

By Hamiltonian function,

$$H = \sum_i P_i\dot{q}_i - L$$

$H = P_\theta\dot{\theta} - L$ [$\because$ θ is only variation l is constant]

where $P_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$.

$\therefore P_\theta = ml^2\dot{\theta}$ .... (1)

$$\therefore H = ml^2\dot{\theta}^2 - \frac{1}{2}ml^2\dot{\theta}^2 + mgl(1 - \cos\theta)$$

$$H = \frac{1}{2}ml^2\dot{\theta}^2 + mgl(1 - \cos\theta)$$
$$= \frac{1}{2}ml^2\left(\frac{P_\theta}{mR}\right)^2 + mgl(1 - \cos\theta) \text{ by } P_\theta \quad \ldots\ldots\ldots(2)$$

$H = T + V$

⇒ The given system is conservative.

From (2), $\frac{\partial H}{\partial P_\theta} = \frac{P_\theta}{ml^2}$

and $\frac{\partial H}{\partial \theta} = mgl\sin\theta$.

Hence by Hamilton equation of motion,

i.e., $\dot{\theta} = \frac{\partial H}{\partial P_\theta} = \frac{P_\theta}{ml^2}$ ....(3)

$\dot{P}_\theta = \frac{-\partial H}{\partial \theta} = -mgl\sin\theta$ .... (4)

By (3), $\dot{P}_\theta = ml^2\ddot{\theta}$

$$\therefore \text{ By (4)} \Rightarrow ml^2\ddot{\theta} = -mgl\sin\theta$$
$$\Rightarrow ml^2\ddot{\theta} + mgl\sin\theta = 0$$
$$\Rightarrow l\ddot{\theta} + g\sin\theta = 0$$

Which is the required equation of motion

Example 3:

Given a mass-spring system consisting of a mass m and a linear spring of stiffness k, as shown in Fig-4. Find the equations of motion using the Hamiltonian procedure. Assume that the displacement x is measured from the unstressed position of the spring.

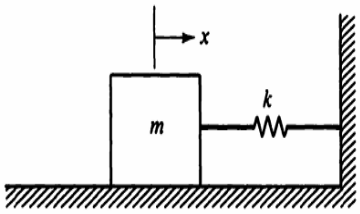


Fig-3.4 A mass-spring system

Let us find the kinetic and potential energies in the usual form

$T = \frac{1}{2}m\dot{x}^2$, $V = \frac{1}{2}kx^2$

$$L = T - V$$
$$= \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

The linear momentum is $P = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$

The kinetic energy in the form $T = \frac{p^2}{2m}$, and the Hamiltonian function is found to be

$H(x, p) = p\dot{x} - L = \frac{p^2}{2m} + \frac{1}{2}kx^2$

Since this is a natural system, the Hamiltonian H is equal to the total energy T+V and is constant.

$\dot{x} = \frac{\partial L}{\partial p} = \frac{p}{m}$, $\dot{p} = -\frac{\partial H}{\partial x} = -kx$

These two first-order equations are equivalent to the single second order equation.

$\Rightarrow m\ddot{x} + kx = 0$

Which is the familiar equation of motion that can be obtained by using Newton's law of motion or Lagrange's equation.

Example: 4

A particle of mass m is attached to a fixed point O. by an inverse square force.

Find the equation of motion.

Proof:

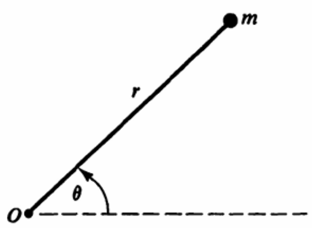


Fig-3.5 The Kepler problem using polar coordinates

Let a particle of mass m is attracted to ' O ' by inverse square force F_r.

$F_r = \frac{-m\mu}{r^2}$, μ is gravitational coefficients.

Let $P(r,\theta)$ be the polar co-ordinate of $P$of mass "m" at time ' t '.

$$\therefore T = \frac{1}{2}m(\dot{r}^2 + r\dot{\theta}^2)$$

and P.E $V = \int \vec{F}_\gamma \cdot dr$.

$$= -\int \left(\frac{-m\mu}{r^2}\right) dr$$

$$= m\mu \int \frac{1}{r^2} d\gamma$$

$$V = \frac{-m\mu}{r}$$

$\therefore$ By Lagrangian function is

$$L = T - V$$

$$= \frac{r}{2}m(\dot{\gamma}^2 + \gamma\dot{\theta}^2) + \frac{m\mu}{\gamma}$$

here generalized momentum are,

$$P_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \Rightarrow \dot{r} = \frac{P_r}{m}$$

$$P_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \Rightarrow \dot{\theta} = \frac{P_\theta}{mr^2}$$

By Hamiltonian function,

$$H = \sum_j P_i\ddot{q}_i - L$$

$$H = P_r\dot{r} + P_\theta\dot{\theta} - \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{m\mu}{r}$$

$$= P_r\left(\frac{P_r}{m}\right) + P_\theta\left(\frac{P_\theta}{mr^2}\right) - \frac{m}{2}\left(\frac{P_r^2}{\mathrm{m}^2} + r^2\left(\frac{P_\theta}{mr^2}\right)^2\right) - \frac{m\mu}{r}$$

$$H = \frac{P_r^2}{m} + \frac{P_\theta^2}{mr^2} - \frac{1}{2}\left(\frac{P_r^2}{m} + \frac{P_\theta^2}{mr^2}\right) - \frac{m\mu}{r}$$

$$= \frac{1}{2}\left(\frac{P_r^2}{m} + \frac{P_\theta^2}{mr^2}\right) - \frac{m\mu}{r}$$

$\therefore$ By Hamiltonian equation of motion are,

$$\dot{r} = \frac{\partial H}{\partial P_r} = \frac{P_r}{m} \Rightarrow \dot{P}_r = m\ddot{r}$$

$$p_{\dot{r}} = -\frac{\partial H}{\partial r} = -\left(-\frac{m\mu}{r^2}\right)$$

$$\Rightarrow m\ddot{r} = \frac{\mathrm{m}\mu}{r^2}$$

$\Rightarrow r^2\ddot{\gamma} - \mu = 0$. which is equation of motion for ' r '

By Hamiltonian equation of motion for θ,

$$\dot{\theta} = \frac{\partial H}{\partial P_\theta} = \frac{P_\theta}{mr^2} \Rightarrow \dot{P}_\theta = mr^2\ddot{\theta}$$

$$\dot{P}_\theta = -\frac{\partial H}{\partial \theta} = 0$$

$\Rightarrow mr^2\ddot{\theta} = 0$

$\Rightarrow \ddot{0} = 0$ which is the required en of motion

Example: 5

Obtain Hamiltonian equation of motion for a charges particle in an electromagnetic field.

Proof:

Let us consider a particle of mass m with charge moving with velocity $\vec{v}$.

Let $\vec{\phi}$ be the Scalar potential.

Let $\vec{A}$ be the vector potential.

$$\therefore T = \frac{1}{2}mv^2$$

$$v = \vec{e}(\vec{\phi} - \vec{v} \cdot \vec{A})$$

$\therefore$ By Lagrangian function is,

$$L = T - V,$$

$$L = \frac{1}{2}mv^2 - \vec{e}(\vec{\phi} - \vec{V} \cdot \vec{A})$$

For generalized momentum,

$$\vec{P}_\theta = \frac{\partial L}{\partial V}$$

$$\overrightarrow{P_v} = m\vec{v} + \vec{e}\vec{A} \Rightarrow \overrightarrow{P_v} - \vec{e}\vec{A} = m\vec{v}$$

$$\Rightarrow \vec{v} = \frac{\overrightarrow{P_v} - \vec{e}\vec{A}}{m}$$

$\therefore$ By Hamiltonian function,

$$H = P_v\vec{v} - L$$

$$= (m\vec{v} + \vec{e}\vec{A})\cdot\vec{v} - \frac{1}{2}mv^2 + \vec{e}(\vec{\phi} - \dot{\vec{v}}\cdot\vec{A})$$

$$= mv^2 + \vec{e}\cdot\vec{A}\cdot\vec{v} - \frac{1}{2}mv^2 + e\bar{\phi} - \vec{e}\cdot\vec{v}/\vec{A}$$

$$H = \frac{1}{2}mv^2 + \bar{e}\bar{\phi}$$

$$= \frac{1}{2}m\left(\frac{P_v - \bar{e}\bar{A}}{m^2}\right)^2 + \vec{e}\phi \quad \text{by } \overrightarrow{P_r}$$

$$H = \frac{1}{2m}(P_v - \bar{e}\bar{A})^2 + \bar{e}\vec{\phi}$$

∴ By Hamiltonian equation of motion for Cartesian co-ordinate.

First (canonical equation).

$$\dot{x} = \frac{\partial H}{\partial P_x} = \frac{1}{m}(P_x - eA_x)$$

$$\dot{y} = \frac{\partial H}{\partial P_y} = \frac{1}{m}(P_y - eA_y)$$

$$\dot{z} = \frac{\partial H}{\partial z} = \frac{1}{m}(P_z - eA_z)$$

$$\Rightarrow \text{ (or) } \vec{V} = \frac{1}{m}(\overrightarrow{P_v} - e\bar{A})$$

Second canonical equation

$$\dot{P}_x = \frac{-\partial H}{\partial x} = -\vec{e}\cdot\frac{\partial\phi}{\partial x} + \frac{e}{m}[(P_x - eA_x)] + \frac{\partial A_x}{\partial x} + (P_y - eAy)\frac{\partial A_y}{\partial x} + (P_z - eA_z)\frac{\partial A_z}{\partial x}]$$

$$\dot{P}_y = -\frac{\partial H}{\partial y} = -\vec{e}\frac{\partial\phi}{\partial y} + \frac{e}{m}\left[(P_x - eA_x)\frac{\partial A_x}{\partial y} + (P_y - eA_y)\frac{\partial A_y}{\partial y} f(P_z - eA_z)\frac{\partial A_y}{\partial y}\right.$$

$$\dot{P}_z = -\frac{\partial H}{\partial z} = -\vec{e}\frac{\partial\phi}{\partial z} + \frac{e}{m}\left[(P_x - eA_x)\frac{\partial A_x}{\partial z} + (P_y - eA_y)\frac{\partial A_y}{\partial z} f(P_z - eA_z)\frac{\partial A_y}{\partial z}\right.$$

These are the required equation of motion?

$$\vec{\vec{P}} = -e\nabla\phi + e\nabla(\vec{v}\cdot\vec{A})$$

Example 6:

Discuss the Kepler's problem writing a Hamiltonian canonical equation of motion.

A particle of mass m is attracted to a fixed point 'o' by an inverse square force.

$\overrightarrow{F_r} = -\frac{m\mu}{r^2}$, where μ is coefficient of gravity. Discuss the motion.

Solution:

Let p(r, θ) be the position of the particle at time t of mass m.

Let $\vec{F}$ be the attractive force in the negative radical direction

∴ The velocity compentent of P is ($\dot{r}$, r$\dot{\theta}$).

$$K = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

$$P \times E = -\int \vec{F}d\vec{r}$$

$$V = -\frac{m\mu}{r^2}dr$$

By Lagrangian function

$$L = T - V$$

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{m\mu}{r}.$$

$$\therefore P_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

$$P_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

∴ By Hamiltonian function,

$$H = \sum P_i\dot{q}_i - L \quad [\because \dot{q}_i = \dot{r} \text{ (or) } \dot{\theta}]$$

$$= P_r\dot{r} + P_\theta\dot{\theta} - \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{m\mu}{r}$$

$$= P_r\left(\frac{P_r}{m}\right) + P_\theta\left(\frac{P_\theta}{mr^2}\right) - \frac{1}{2}m\{\frac{P_r^2}{m^2} + r^2P_\theta^2\}$$

$$H = \left(\frac{P_r^2}{m} + \frac{P_\theta^2}{mr^2}\right) - \frac{1}{2}\cdot\frac{P_r^2}{m} - \frac{1}{2}\frac{P_\theta^2}{mr^2} - \frac{m\mu}{r}$$

$$H = \frac{1}{2}\left(\frac{P_r^2}{m} + \frac{P_\theta^2}{mr^2}\right) - \frac{m\mu}{r}.$$

∴ By Hamilton canonical Equation,

$$\dot{P}_i = -\frac{\partial H}{\partial q_i}$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad , \dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \quad \ldots\ldots\ldots (1)$$

$$\dot{P}_\gamma = -\frac{\partial H}{\partial \mathrm{r}} = \frac{1}{2}\cdot p_\theta^2\cdot\left(\frac{-2}{mr^3}\right) + \frac{m\mu}{r^2}.$$

$$\dot{P}_\theta = -\frac{\partial H}{\partial \theta} = 0.$$

For " r ",

By (1) $\Rightarrow m\ddot{r} = \dot{P}_r = \frac{P_0^2}{mr^3} + \frac{m\mu}{r^2}$.

For " θ "

By (1) $\Rightarrow mr^2\ddot{\theta} = \dot{p}_\theta$

$mr^2\ddot{\theta} = 0$

(i.e.,) $\ddot{\theta} = 0$

These are the equation of motion.

3.3. Other Variational Principles:

The varied paths are taken in an n-dimensional configuration spaces and are restricted by the conditions that the δq's conform to the instantaneous constraints, if any and vanish at the end-points. Furthermore, the $\delta\dot{q}$'s are related to the δq's by the equations

$$\delta\dot{q}_i = \frac{d}{dt}(\delta q_i)$$

They $\delta\dot{q}$'s are not zero, in general, at the end-points

Theorem 1:

State and prove modified Hamilton principle

The actual path is such that integral of equation. $\delta\int_{t_0}^{t}(\sum_{i=1}^{n} p_i\dot{q}_i - H)dt = 0$ is stationary for arbitrary variation of the path is phase space, with the restriction that the δq's vanish at first times t_0 & t_1. The δp's need not be zero at these points.

Proof:

By Hamiltonian principle.

$$\delta\int_{t_0}^{t_1} Ldt = 0$$

where the end points corresponding to to and, are fixed, and $\delta q's$ varishos at end pts. and $\delta\dot{q}_i = \frac{d}{dt}(\delta q_i)$

$\delta\dot{q}_i \neq 0$ at t_0, t_1

(i.e.,) there is a no variation of time with respect to virtual displacement.

But $H = \sum P_i\dot{q}_i - L$

$$L = \sum_{i=1}^{n} P_i\dot{q}_i - H$$

$\therefore \delta\left(\int_{t_0}^{t} \left(\sum_{i=1}^{n} P_i\dot{q}_i - H\right)dt\right) = 0$

$\int_{t_0}^{t_1} \left(\sum_{i=1}^{n} p_i\delta\dot{q}_i + \dot{q}_i\delta p_i - \frac{\partial H}{\partial q_i}\delta q_i - \frac{\partial H}{\partial p_i}\delta p_i\right)dt = 0$ .... (2) [H=H(q, p)]

But $\int_{t_0}^{t_1} p_i\delta\dot{q}_i\, dt = \int_{t_0}^{t_1} p\, d(\delta q_i)$

$= p[\delta q_i]_{t_0}^{t_1} - \int_{t_0}^{t} (\delta q_i)\frac{dp_i}{dt}dt.$

$\therefore \int_{t_0}^{t_1} p\delta\dot{q}_i dt = 0 - \int_{t_0}^{t_1} (\delta q_1)\dot{p}_i dt$

$$(2)\ \int_{t_0}^{t_1} \sum_{i=1}^{n} \left(-\dot{p}\delta q_i + \dot{q}_i\delta p_i - \frac{\partial H}{\partial q_i}\delta q_i - \frac{\partial H}{\partial p_i}\delta p_i\right)dt = 0$$

(i.e.) $\int_{t_0}^{t_1}\sum_{i=1}^{n} \left(\delta p_i\left(\dot{q}_i - \frac{\partial H}{\partial p_i}\right) + \delta q_i\left(-\dot{p}_i - \frac{\partial H}{\partial q_i}\right)\right] dt = 0$ ....(3)

By equation (1)

$\delta\dot{q}_i = \frac{d}{dt}(\delta q_i)$

we find that $\delta\dot{q}_i$ (or) δp_i is not independent of δq_i.

(i.e.,) δ p's and $\delta q's$ are dependent.

But from value of H.

$$\left.\begin{aligned}\dot{q}_i &= \frac{\partial H}{\partial p_i}\\ \dot{p}_i &= -\frac{\partial H}{\partial q_i}\end{aligned}\right\}\quad \text{canonical form of Hamiltonian.}$$

$\therefore$ Hence eqn (3) will be true.

Hence the modified Hamilton principle is of the form,

$\delta\left(\int_{t_0}^{\pm}\left(\sum_{i=1}^{n} P_i\dot{q}_i - H\right)dt\right) = 0$ holds good for the phase Space.

Theorem 2:

State and prove principle of Least action.

The actual path of a conservative holonomic system is such that the action is stationary with respect to varied path having the same energy integral h and the same and points in the q-space.

(i.e.,) $\delta A = \delta\int_{t_0}^{t_1}\sum_{i=1}^{n} p_i\dot{q}_i dt = 0.$

Proof:

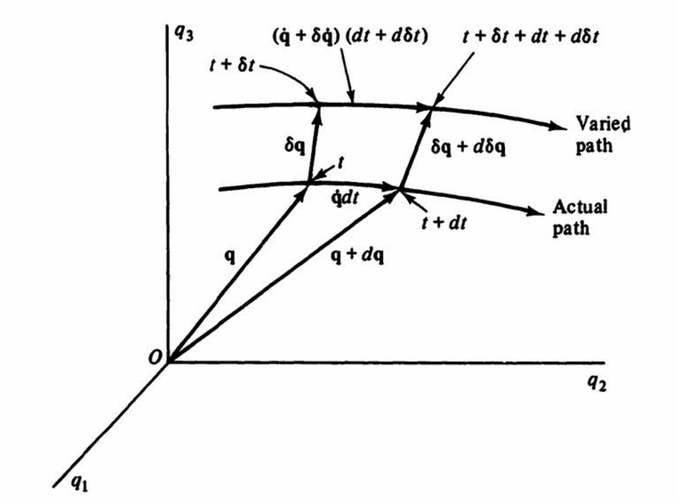


Fig 3.6. A general variation in configuration space

A general variation in configuration space.

Let us consider the small quadrilateral as in the figure

we have,

$$\dot{q}dt + \delta\vec{q} + d\delta\vec{q} = \delta\vec{q} + (\dot{q} + \delta\dot{q})(dt + d\delta t)$$
$$= \delta\vec{q} + \dot{q}dt + \dot{q}d(\delta t) + \delta\dot{q}dt + \delta\dot{q}d(\delta t)$$
$$\Rightarrow d(\delta\vec{q}) = \dot{\vec{q}}d(\delta t) + \delta\dot{\vec{q}}dt + \delta\dot{\vec{q}}d(\delta t).$$
$$\Rightarrow \frac{d}{dt}(\delta\vec{q}) = \dot{\vec{q}}\frac{d}{dt}(\delta t) + \delta\dot{\vec{q}} + \delta\dot{\vec{q}}\frac{d}{dt}(\delta t))$$
$$\Rightarrow \delta\dot{\vec{q}} = \frac{d}{dt}(\delta\vec{q}) - \dot{\vec{q}}\frac{d}{dt}(\delta t) \quad \ldots\ldots\ldots\ldots(1)$$

Let us define an integral I as,

$$I = \int_{t_0}^{t_1} L(q,\dot{q},t)dt$$
$$\Rightarrow \delta I = \delta\int_{t_0}^{t_1} L(q,\dot{q},t)dt.$$
$$\Rightarrow \delta I = \int_{t_0}^{t_1}\left[\delta L + L\frac{d}{dt}(\delta t)\right]dt$$
$$= \int_{t_0}^{t_0^1}\left[\sum_{i=1}^{n}\left(\frac{\partial L}{\partial\dot{q}_i}\delta\dot{q}_i + \frac{\partial L}{\partial q_i}\delta q_i\right) + \frac{\partial L}{\partial t}\delta t \ + L\frac{d}{dt}(\delta t)\right]dt \quad [\because \text{eqn}(1)]$$
$$\delta I = \int_{t_0}^{t_0^1}\left[\sum_{i=1}^{n}\left(\frac{\partial L}{\partial\dot{q}_i}(\frac{d}{dt}\delta q_i - \dot{q}_i\frac{d}{dt}\delta t)\right) + \frac{\partial L}{\partial q_i}\delta q_i \ + \frac{\partial L}{\partial \mathrm{t}}\delta\mathrm{t} + L\frac{d}{dt}(\delta t)\right]dt \quad \ldots\ldots\ldots\ldots..(2)$$

$$= \frac{d}{dt}\left(\sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\delta q_i\right) = \sum_{i=1}^{n} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i + \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}(\delta q_i)$$

$$\Rightarrow \sum_{i=1}^{N} \frac{\partial L}{\partial q_i}\frac{d}{dt}(\delta q_i) = \frac{d}{dt}\left(\sum_{i=1}^{h} \frac{\partial L}{\partial \dot{q}_i}\delta q_i\right) - \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i\right) \qquad \ldots\ldots\ldots\ldots (3)$$

sub (3) in (2)

$$\delta I = \int_{i_0}^{i_1}\left[\frac{d}{dt}\left(\sum_{i=1}^{n} \frac{\partial L}{\partial q_i}\delta q_i\right) - \sum_{i=1}^{n} \frac{d}{dt}\left(\frac{\partial L}{\partial i_i}\right)\delta q_i\right]$$

$$-\left[\sum_{i=1}^{n} \frac{\partial L}{\partial q_i}\dot{q}_i + \frac{d}{dt}(\delta t) + \sum_{i=1}^{n} \frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial t}\delta t + L\frac{d}{dt}(\delta t)\right.$$

$$\delta I = \int_{t_0}^{t_1} \frac{d}{dt}\left(\sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\delta q_i\right)dt + \int_{t_0}^{t_1}\left[\frac{\delta L}{\partial t}dt - \left(\sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L\right)\frac{d}{dt}(\delta t)\right]dt$$

$$-\int_{t_0}^{t_1^1} \sum_{i=1}^{n}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i}\right]\delta q_i dt$$

$$= 0 + \int_{i_0}^{t_1}\left[\frac{\partial L}{\partial t}\delta t - \left(\sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L\right)\frac{d}{dt}(\delta t)\right]dt - 0$$

[$\because \delta q_i = 0$ at c and points and by L Eqn

$$\Rightarrow \delta I = \int_{I_0}^{1}\left[-\sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i + L\right]\frac{d}{dt}(\delta t)\right]dt$$

[$\because$ for conservative system $\frac{\partial L}{\partial t} = 0$]

$$= \int_0^{t_1} -n\frac{d}{dt}\delta t dt$$

$$= -h\int_{t_0}^{t_1} d(\delta t)$$

t_0

$$\delta I = -h[\delta t]_{t_0}^{t_1}$$

But action is given by,

$$A=\int_{t_0}^{t_1}\sum_{i=1}^{n}\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i dt$$

$$A=\int_{t_0}^{t_1}\sum_{i=1}^{n}\frac{\partial L}{\partial q_i}q_i dt.$$

$$\delta A=\delta\int_{t_0}^{t_1}(L+h)dt \text{ [by result *]}$$

$$=\int_{t_0}^{t_1}\delta(L+h)dt$$

$$=\int_{t_0}^{t_1}\delta L dt+\int_{t_0}^{t_1}\delta(hdt)$$

$$=\delta I+\int_{t_0}^{t_1}[\delta h dt+hd(\delta t)]$$

$$=\delta I+\delta(ht)t_0+h[\delta t]_{t_0}^{t_1}$$

$$\because \int_{t_0}^{t_1} dLdt=dI$$

$$=-h[\delta t]_{t_0}^{t_1}+\delta h[t_1-t_0]+h[\delta t]_{t_0}^{t_1}$$

$$\delta A=\delta h(t_1-t_0)$$

$$\delta A=0 \ \because \delta h=0$$

$\Rightarrow A$ is stationary value

Example 1:

Express Jacobi form of principle of Least action

Proof:

By principle of least action,

$\Rightarrow \delta A=0$

$\Rightarrow \delta\int_{t_0}^{t_1} 2\text{T}\, dt=0$

$\Rightarrow \delta\int_{t_0}^{t_1} 2\sqrt{T^2}dt=0$

$$\Rightarrow \delta\int_{t_0}^{t_1} 2\sqrt{TT}dt$$

$$\Rightarrow \delta\int_{t_0}^{t_1} 2\sqrt{T(h-v)}dt=0 \quad \ldots\ldots\ldots\ldots (1)$$

(1) [$\because$ for natural system $T+v=h$]

But, $ds^2=\sum_{i=1}^{n}\sum_{j=1}^{n}m_{ij}\dot{q}_i\dot{q}_j dt^2$.

$= \sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij} \frac{dq_i}{dt}\frac{dq_j}{dt}$ dt dt.

$ds^2 = 2Tdt^2$

$ds = \sqrt{2\text{T}}\, dt$ (or) $dt = \frac{ds}{\sqrt{2\text{T}}}$

Sub in (1)

$$\delta A = \delta \int_{t_0}^{t_1} 2\sqrt{T(h-v)} \frac{ds}{\sqrt{2\text{T}}} = 0$$

$$\delta A = \delta \int_{t_0}^{t_1} 2\sqrt{T(h-v)} ds = 0$$

This is the Jacobi form of principle of least action.

Example 2:

Establish the Jacobian Integral equation [or] Equation of Energy by-using Hamiltonian. Canonical equation.

Proof:

Since T is a homogeneous quadratic fuctions are $\dot{q}_1, \dot{q}_2$ ...

By Euler's theorem,

$$\sum_{i=1}^{n} \frac{\partial T}{\partial \dot{q}_i} \dot{q}_i = 2T$$

But $L = T - V$

$$\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} - 0.$$

$\sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i = 2T$ ………….. (1)

But $H = H(P_i, q_i)$.

$$\frac{\partial H}{\partial t} = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial p_i} \dot{p}_i + \frac{\partial H}{\partial q_i} \dot{q}_i \right)$$

$= \sum_{i=1}^{n} -\left(\frac{\partial H}{\partial p_i} \cdot \frac{\partial H}{\partial q_i} + \frac{\partial H}{\partial q_i} \cdot \frac{\partial H}{\partial p_i} \right)$ [∵ by Hamilton canonical eqn]

$$\frac{\partial H}{\partial t} = 0.$$

Integrate partially w.r.t.

$H =$ constant $= h$

$$\Rightarrow \sum_{i=1}^{n} p_i \dot{q}_i - L = h.$$

$$\Rightarrow \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L = h$$

$\Rightarrow 2T - (T - v) = h$

$\Rightarrow T + V = h$

$\Rightarrow \therefore$ Total energy is constant.

Unit IV:

Hamilton-Jacobi Theory: Hamilton's Principle function – Hamilton-Jacobi Equation – Separability.

Chapter 4: Sections 4.1-4.3

Introduction:

In this chapter we shall approach the problem by studying the generating function which is associated with the required canonical transformation. The generating function is the solution of a partial differential equation known as the Hamilton-Jacobi Equation. The transformation equations, and hence the solution to the problem, are obtained from the generating function by a process of differentiations and algebraic manipulation.

4.1. Hamilton's principle Function:

Theorem 1:

Derive Hamilton principle Function.

Proof:

The canonical Integral: Consider $I = \int_{t_0}^{t_1} L dt$

We have the 2n initial conditions of $P_1, P_2 \ldots P_2$ and $q_1, q_2 \cdots q_n$ that is equal to,

$q_{i0}(i = 1 + 0n)$, $p_{i0} = (i = 1 \text{ to n})$

Here t_0 and t_1 are the end points of the trajectory in the q-space are fixed

Let $\dot{q}_{io} = \eta_i(\dot{q}_0, \dot{q}_1, t_0, \dot{f}_1)$

Where we consider t, as running time

Let $I = \int_{t_0}^{t_1} L dt$ be evaluated as a function of $(q_{i0}, \dot{q}_{i0}, t_0, t_1)$.

By using the relation of $\dot{q}_{i0} = \eta_i(\dot{q}_0, \dot{q}_1, t_0, t_1)$ hence we get, $I = \int_{t_0}^{t_1} L dt = S(q_0, q_1, t_0, t)$

This S is called Hamilton principle function. Now consider,

$$\delta I = \int_{t_0}^{t_1} \sum_{i=1}^{n} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \delta q_i dt + \int_{t_0}^{t_1} \frac{\partial L}{\partial t} \delta t$$

$$\delta I = \int_{t_0}^{t_1} \sum_{i=1}^{n} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \delta q_i dt + \int_{I_0}^{t_1} \frac{\partial L}{\partial t} \delta t - \left(\sum_{i=1}^{1} \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - \mathrm{L} \right)$$

$$\left. \frac{d}{dt} \delta t \right] dt - \int_{t_0}^{t_1} \sum_{i=1}^{n} \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} \right] \delta q_i dt \quad \ldots\ldots\ldots (1)$$

Since we consider then standard holonomic system, Lagrangian equations are satisfied.

∴ The third integral vanishes,

Also, $H = H(q_i, p_i, t)$.

$$\therefore H = H(q_i, p_i, t)$$

$$\dot{H} = \frac{dH}{dt} = \sum_i \frac{\partial H}{\partial q_i} \dot{q}_i + \sum_i \frac{\partial H}{\partial P_i} \dot{P}_i + \frac{\partial H}{\partial t}$$

$$\Rightarrow \dot{H} = -\sum_{i=1}^{n} \dot{P}_i \dot{q}_i + \sum \dot{q}_i \dot{P}_i + \frac{\partial H}{\partial t}$$ [by Hamilton canonical equation

$$\Rightarrow \dot{H} = \frac{\partial H}{\partial t} = \frac{-\partial L}{\partial t}.$$

$$(1) \Rightarrow \delta I = \int_{t_0}^{t} \sum_{i=1}^{n} \left[\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \delta q_i \right] dt + \int_{t_0}^{t} \left[-\frac{\partial H}{\partial t} \delta t \right] - H \frac{d}{dt}(\delta t) \Big] dt$$

$$= \int_{t_0}^{t_i} \sum_{i=1}^{n} \frac{d}{dt}(p_i \delta q_i) - \frac{d}{dt}(H \delta t) \Big] dt \qquad \left[\because p_i = \frac{\partial L}{\partial \dot{q}_i} \right]$$

$$= \int_{t_0}^{t_1} \frac{d}{dt} \left[\sum_{i=1}^{n} p_i \delta q_i - H \delta t \right] dt$$

$$\therefore \delta S = \left[\frac{d}{dt} p_i \cdot \delta q_i - \mathrm{H}\, \delta t \right]_{t_0}^{t_1}$$

$$\delta S = \left[\sum_{i=1}^{n} p_i \delta q_{i1} - H_1 \delta t_1 \right] - \left[\sum_i p_{i0} \delta q_{i0} - H_0 \delta t_0 \right]$$

$$\delta S = \sum_i p_{i1} \delta q_{i1} - \sum_i p_{i0} \delta q_{i0} - H_1 \delta t_1 + H_0 \delta t_0$$

Now, $s = s(q_0, q_1, t_0, t_1)$

$$\therefore \delta S = \sum_i \frac{\partial S}{\partial q_{i1}} \delta q_{i1} + \sum_i \frac{\partial S}{\partial q_{i0}} \delta q_{i0} + \frac{\partial S}{\partial t_1} \delta t_1 + \frac{\partial S}{\partial t_0} \delta t_0 \Big)$$

Since we assume that both the variation in S are equal and that $2n + 1$ arguments varied independently, ∴ we get

$$P_{i1} = \frac{\partial s}{\partial q_{i1}}$$
$$P_{i0} = \frac{\partial s}{\partial q_{i0}}, \qquad i = 1,2 \ldots - n$$
$$H_1 = \frac{-\partial s}{\partial t_1}$$
$$H_0 = \frac{-\partial s}{\partial t_0}$$

If we assume that $\frac{\partial^2 s}{\partial q_{i0}\, \partial q_{i1}} \neq 0.$

We can solve for $q_{i1} = q_{i1}(q_0, p_0, t_0, t_1)$, $i = 1$ to n,

(i.e.,) each of the generalized co-ordinate are as a function of current time t_1,

Hence Hamilton principle function is Known to reach the complete solution Stage.

Pfaffian differential form:

Definition:

Consider the m variables $x_1, x_2 \cdots x_n$.

then $\Omega = x_1(x_1, x_2 \ldots x_m)dx_1 + x_2(x_1 - \cdots x_m)dx_2 + \cdots \ldots \ldots + x_m(x_1, x_2 - \cdots x_m)dx_m$.

is called Pfaffian differential equation.

Theorem 1:

Derive the Pfaffian differential form:

If we consider a η dimensional space, the pfaffian differential form lead to a line integral over a path in this n-dimensional space.

Let $C_{ij} = \frac{\partial x_i}{\partial x_j} - \frac{\partial x_j}{\partial x_i}$

The condition for the Pfaffian Ω to be a perfect differential is that each of c_{ij} 's should be zero.

[Pfaffian form is exact differential $j \Rightarrow c = 0$] Consider, δs

$$\delta S = \left(\sum_i p_{i1} dq_{i1} - \sum_i p_{i0} dq_{i0}\right) - (H_1 dt_1 - H_0 dt_0)$$

[as above bookwork]

$$= \left[\sum_i p_{i1} \delta q_{i1} - H_1 dt\right] - \left[\sum_i p_{i0} \delta q_{i0} - H_0 dt_0\right]$$
$$= \Omega_1 - \Omega_2.$$

is, the difference of two Pfaffian form,

$$\therefore \Omega = \sum_{i=1}^{n} p_i dq_i - Hdt$$
$$=(p_1 dq_1 + p_2 dq_2 + \cdots + p_n dq_n) + (odp_1 + odp_2 + \cdots + odp_n) - Hdt$$

If ' m ' is odd. There is an associated System of differential equation is given by,

$$\sum_{i=1}^{m} c_{ij} dx_j = 0 \ (j = 1 \text{ fo } n)$$

This is called first Pfaffian form.

In the current discussion we have $2n + 1$ variables namely nq's, np's and t have such a relation exist.

we get, $$\left.\begin{aligned} dq_j - \frac{\partial H}{\partial p_j} dt = 0 \\ -dp_j - \frac{\partial H}{\partial q_j} dt = 0 \end{aligned}\right\} \qquad \ldots\ldots\ldots..(1)$$

and $\sum_{j=1}^{n} \left[\frac{\partial H}{\partial q_j} dq_j + \frac{\partial H}{\partial p_j} dp_j\right] = 0$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$
$$\dot{q}_i = -\frac{\partial H}{\partial p_i}$$

Here equation (1) are Hamiltonian Canonical equation,

$$\Rightarrow \dot{H} = \frac{dH}{dt} = \sum_j \frac{\partial H}{\partial q_j} \dot{q}_j + \sum_j \frac{\partial H}{\partial p_j} \dot{p}_j + \frac{\partial H}{\partial t}$$
$$= \sum_j \left[\frac{\partial H}{\partial q_j} \cdot \frac{\partial H}{\partial p_j} - \frac{\partial H}{\partial p_j} \cdot \frac{\partial H}{\partial q_j}\right] + \frac{\partial H}{\partial t}$$
$$\Rightarrow \dot{H} = \frac{\partial H}{\partial t}.$$

4.2. The Hamilton-Jacobi Equation:

Theorem 1:

Derive Hamilton Jacobi equation:

$$I = \int_{t_0}^{t_1} Ldt$$

we have $2n$ initial conditions $q_{10}, q_{20'} \ldots q_{n0}$

$P_{10}, P_{20}, \dots P_{n0}$.

Here t_1 as running time.

$\therefore I = \int_{t_0}^{t_1} L dt$ can be evaluated as a function of $(q_{10}, \dot{q}_{10}, t_0, t_1)$, $(i = 1 \text{ to } n)$

Let $\delta = \int_{t_0}^{t_1} L dt = I = \delta(q_0, \dot{q}_0, t_0, t_1)$

[By Hamilton principle function]

Using expression for δI,

we get,

$$ds = \sum_{i=1}^{n} P_{i1} dq_{i1} - \sum_{i=1}^{n} P_{i0} dq_{i0} - H_1 dt_1 - H_0 dt_0.$$

Let $2n$ initial conditions are given by,

$$\alpha = \alpha_i(q_{10}, q_{20} - q_{n0}, p_{10}, p_{20} - \cdots p_{n0}) \text{ and } \beta = \beta_i(q_{10}, q_{20} - q_{n0}, p_{10}, p_{20} \cdots p_{n0})$$

$$\sum_{i=1}^{n} p_{i0} dq_{i0} = \sum_{i=1}^{n} \beta_i d\alpha_i \ (i = 1 \text{ to } 2n).$$

consider, $S = S(q_{i0}, \dot{q}_{i0}, 1_0, 1)$ which can be written as,

$S = S(q_i, \alpha_i, t_0, t_1)$

now, $ds = \sum_{i=1}^{n} \frac{\partial s}{\partial q_{i1}} dq_{i1} + \sum_{i=1}^{n} \frac{\partial s}{\partial \alpha_1} d\alpha_i + \frac{\partial s}{\partial t_1} dt_1 + \frac{\partial s}{\partial t_0} dt_0$

But $ds = \sum_{i=1}^{n} P_{i1} dq_{i1} - \sum_{i=1}^{n} P_{i0} dq_{i0} - H_1 dt_1 - H_0 dt_0$

$ds = \sum_{i=1}^{n} P_{i1} dq_{i1} - \sum_{i=1}^{n} \beta_i d\alpha_i - H_1 dt_1 - H_0 dt_0$ .... (1)

Assuming that the variation to be independent we get,

$$P_{i1} = \frac{\partial s}{\partial q_{il}}$$

$$\beta_i = -\frac{\partial s}{\partial \alpha_i}$$

$$H_1 = -\frac{\partial s}{\partial t_1}$$

$$H_0 = -\frac{\partial s}{\partial t_0}$$

So Long, we using to initial value of time and t, as its value at any instance:

Then for simplification,

Let us make $t_0 = 0$ and replace t, by t.

we get,

$ds = \sum p_i dq_i - \sum \beta_i d\alpha_i - Hdt$ by (1).

$\therefore S$ is of the form $S(a, \alpha, 1)$

$$\therefore ds = \sum_i \frac{\partial s}{\partial q_i} dq_i + \sum_i \frac{\partial s}{\partial x_i} d\alpha_i + \frac{\partial s}{\partial t} dt$$

hence, we get

$$P_i = \frac{\partial s}{\partial q_i}$$

$$\beta_i = -\frac{\partial s}{\partial \alpha_i}$$

$$H = -\frac{\partial s}{\partial t}$$

$$\therefore \frac{\partial s}{\partial t} + H = \frac{\partial s}{\partial t} + H(q, p, t) = 0$$

(i.e.,) $\frac{\partial s}{\partial t} + H\left(q, \frac{\partial s}{\partial q}, t\right) = 0.$

This is called Hamilton Jacobi equation.

Theorem 2:

State and prove Jacobi's theorem:

Statement:

If $\delta(q, \alpha, t)$ is any complete Solution of Hamilton Jacobi equation,

$\frac{\partial S}{\partial t} + H\left(q, \frac{\partial s}{\partial q}, t\right) = 0$ and if

$$P_i = \frac{\partial s}{\partial q_i}$$

$$\beta_i = -\frac{\partial s}{\partial \alpha_i}, \ i = 1 \text{ to } n.$$

where $\beta_i' s$ are arbitrary constant, are used to solve for $q_i(\alpha, \beta, t)$ and $p_i(\alpha, \beta, t)$

Then these expressions also provide the general Solution of canonical equation of Hamilton.

Proof:

By Hamilton Jacobi equation, $\frac{\partial s}{\partial t} + H\left(q, \frac{\partial s}{\partial q}, t\right) = 0$ ……….(1)

Diff partially (1) w.r.t α_i

$$\frac{\partial^2 S}{\partial \alpha_i \, \partial t} + \sum_j \frac{\partial H}{\partial p_j}\frac{\partial p_j}{\partial \alpha_i} = 0 \qquad \left[\text{here } p_j = \frac{\partial s}{\partial q_j}\right]$$

Again,

$$\frac{\partial^2 S}{\partial \alpha_i \, \partial t} + \sum_i \frac{\partial H}{\partial p_j}\frac{\partial^2 S}{\partial \alpha_i \, \partial q_j} = 0 \;\ldots\ldots\ldots\ldots(2) \quad \left[\frac{\partial p_j}{\partial \alpha_i} = \frac{\partial^2 S}{\partial \alpha_i \, \partial q_j}\right]$$

Also, $-\beta_i = \frac{\partial s}{\partial \alpha_i}, i = 1, \ldots n.$

Diff partially w.r.t " t"

$$\Rightarrow 0 = \frac{\partial^2 S}{\partial t \, \partial \alpha_i} + \sum_{j=1}^{n} \frac{\partial^2 S}{\partial q_j \, \partial \alpha_i} q_j = 0 \;\ldots\ldots\ldots\ldots.(3)$$

(3) -(2)

$$\Rightarrow \sum_j \left(\dot{q}_j - \frac{\partial H}{\partial p_j}\right)\frac{\partial^2 S}{\partial q_j \, \partial \alpha_i} = 0.$$

$$\Rightarrow \dot{q}_j = \frac{\partial H}{\partial p_j}, (j = 1 \ldots \ldots n) \qquad \left[\because \frac{\partial^2 s}{\partial q_j \, \partial \alpha_i} \neq 0 \text{ by assumption}\right]$$

Again consider (1),

Diff partially (1) w.r.t " q_j".

$$\frac{\partial^2 s}{\partial q_j \, \partial t} + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\frac{\partial p_i}{\partial q_j} + \frac{\partial H}{\partial q_j} = 0$$

$$\frac{\partial^2 s}{\partial q_j \, \partial t} + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} \cdot \frac{\partial^2 S}{\partial q_i \, \partial q_j} + \frac{\partial H}{\partial q_j} = 0 \qquad \ldots\ldots\ldots\ldots (4)$$

$$p_j = \frac{\partial S}{\partial q_j}$$

Diff partially w.r.t " t ".

$$\dot{p}_j = \frac{\partial^2 s}{\partial t \, \partial q_j} + \sum_{i=1}^{n} \frac{\partial^2 s}{\partial q_i \partial q_j} \cdot \frac{\partial H}{\partial p_i} = 0 \qquad \ldots\ldots\ldots\ldots (5)$$

$$(5) + (4) \Rightarrow \dot{P}_j + \frac{\partial H}{\partial q_j} = 0$$

$$\Rightarrow \dot{p}_J = -\frac{\partial H}{\partial q_J} \; (j = 1,2, \ldots, n)$$

which is the second canonical equation. Thus we see that any complete solution of the Hamilton-Jacobi equation leads to a solution of the Hamilton problem. This solution has the proper number of arbitrary constants and, of course, obeys the canonical equations.

Conservative Systems and Ignorable Coordinates:

Now consider a conservative holonomic system whose configuration is described in terms of n independent q 's. The Hamiltonian function for this system is not an explicit function of time and, in fact, is a constant of the motion.

$$H(q,p) = \alpha_n = h \quad \ldots\ldots\ldots\ldots (1)$$

where h is the value of the familiar Jacobi integral or energy integral which we arbitrarily identify with α_n.

A suitable form for the principal function of this system is found by using Equation (1) and

Hamilton Jacobi equation $\frac{\partial S}{\partial t} + H\left(q, \frac{\partial s}{\partial q}, t\right) = 0$

to obtain $\frac{\partial S}{\partial t} = -H = -\alpha_n \quad \ldots\ldots\ldots\ldots(2)$

This suggests that S can be taken as a linear function of time, that is,

$$S(q,\alpha,t) = -\alpha_n t + W(q,\alpha) \quad \ldots\ldots\ldots\ldots (3)$$

where we have omitted an arbitrary additive constant. The function $W(q_1,\ldots,q_n,\alpha_1,\ldots,\alpha_n)$ does not contain time explicitly and is known as the characteristic function.

$$\frac{\partial S}{\partial \alpha_t} = \frac{\partial W}{\partial \alpha_t} \qquad (i = 1,2,\ldots,n-1) \quad \ldots\ldots\ldots\ldots\ldots (4)$$

$$\frac{\partial S}{\partial \alpha_n} = \frac{\partial W}{\partial \alpha_n} - t \quad \ldots\ldots\ldots\ldots\ldots (5)$$

$$\frac{\partial S}{\partial q_i} = \frac{\partial W}{\partial q_i} \qquad (i = 1,2,\ldots,n) \quad \ldots\ldots\ldots\ldots\ldots (6)$$

From Equations (2) and (6) we see that the Hamilton-Jacobi equation reduces to

$$H\left(q, \frac{\partial W}{\partial q}\right) = \alpha_n \qquad \ldots\ldots\ldots\ldots (7)$$

Equation (7) is the modified Hamilton-Jacobi equation. A complete solution of this equation involves $(n-1)$ nonadditive α 's, $(\alpha_1,\alpha_2,\ldots,\alpha_{n-1})$, plus the energy constant α_n. The α 's are arbitrary in the sense that their values are determined by the arbitrary initial values of the $\partial W/\partial q_i$, that is, the generalized momenta, at the given initial configuration.

A comparison of Equation (4) & (6) with equations $P_i = \frac{\partial s}{\partial q_i}$, $\beta_i = -\frac{\partial s}{\partial \alpha_i}$

shows that the solution of the Hamilton problem can be obtained from

$$-\beta_i = \frac{\partial W}{\partial \alpha_i} \; (i = 1,2,\ldots,n-1) \quad \ldots\ldots\ldots\ldots (8)$$

$$t - \beta_n = \frac{\partial W}{\partial \alpha_n} \quad \ldots\ldots\ldots\ldots\ldots\ldots (9)$$

$$p_i = \frac{\partial W}{\partial q_i} \; (i = 1,2,\ldots,n) \; \ldots\ldots\ldots\ldots (10)$$

where β_n is the initial time t_0. Since W is not an explicit function of time, we see that Equation (8) gives the path of the system in configuration space without reference to time. Equation (9) then gives the relation of time to position along the path.

Now suppose we consider a system having ignorable coordinates $q_1, q_2, \ldots, q_k$. Initially we shall assume that the system is not conservative. We know that the p 's associated with the ignorable q 's are constant; hence we can take

$$p_t = \alpha_t \; (i = 1,2,\ldots,k)$$

Then we see from Eq. (5-55) that we can assume a principal function of the form

$$S(q,\alpha,t) = \sum_{t=1}^{k} \alpha_i q_t + S'(q_{k+1},\ldots,q_n,\alpha_1,\ldots,\alpha_n,t)$$

The Hamilton-Jacobi equation leads in this case to

$$\frac{\partial S'}{\partial t} + H\left(q_{k+1},\ldots,q_n,\alpha_1,\ldots,\alpha_k,\frac{\partial S'}{\partial q_{k+1}},\ldots,\frac{\partial S'}{\partial q_n},t\right) = 0$$

The complete solution of this partial differential equation involves $(n-k)$ nonadditive constants, exclusive of the constant momenta $\alpha_1, \alpha_2, \ldots, \alpha_k$. Once S' is known, the solution for the motion of the system is obtained from

$$-\beta_i = q_t + \frac{\partial S'}{\partial \alpha_t} \; (i = 1,2,\ldots,k)$$

$$-\beta_t = \frac{\partial S'}{\partial \alpha_t} \; (i = k+1, \dots, n)$$

$$p_t = \alpha_t \; (i = 1,2, \dots, k)$$

$$p_t = \frac{\partial S'}{\partial q_t} \; (i = k+1, \dots, n)$$

where we note that $\beta_i = -q_{i0}$ $(i = 1,2, \dots, k)$

that is, each β corresponding to an ignorable coordinate is just the negative of the initial value of this coordinate.

Finally, let us consider a system which has ignorable coordinates q_1, $q_2, \dots, q_k$ and is also conservative. Combining the previous results, we see that the principal function has the form

$$S(q, \alpha, t) = \sum_{i=1}^{k} \alpha_t q_t - \alpha_n t + W'(q_{k+1}, \dots, q_m \alpha_1, \dots, \alpha_n)$$

and the modified Hamilton-Jacobi equation becomes

$$H\left(q_{k+1}, \dots, q_n, \alpha_1, \dots, \alpha_k, \frac{\partial W'}{\partial q_{k+1}}, \dots, \frac{\partial W'}{\partial q_n}\right) = \alpha_n$$

The complete solution for W' in this case involves the $(n-k-1)$ nonadditive constants $\alpha_{k+1}, \dots, \alpha_{n-1}$ plus, of course, the energy constant α_n and the constant momenta $\alpha_1, \alpha_2, \dots, \alpha_k$.

The motion of the system is given by

$$-\beta_i = q_t + \frac{\partial W'}{\partial \alpha_i} \; (i = 1,2, \dots, k)$$

$$t - \beta_n = \frac{\partial W'}{\partial \alpha_n}$$

$$p_i = \alpha_i \; (i = 1,2, \dots, k)$$

$$p_i = \frac{\partial W'}{\partial q_i} \; (i = k+1, \dots, n)$$

Example 1:

As a first illustration of the Hamilton-Jacobi method, consider its application to a simple mass-spring system (Fig. 5-1). This is a natural system having kinetic and potential energies given by $T = \frac{1}{2}m\dot{x}^2,\ V = \frac{1}{2}kx^2$ ....(1)

The momentum is the familiar $p = \frac{\partial T}{\partial \dot{x}} = m\dot{x}$ ....(2)

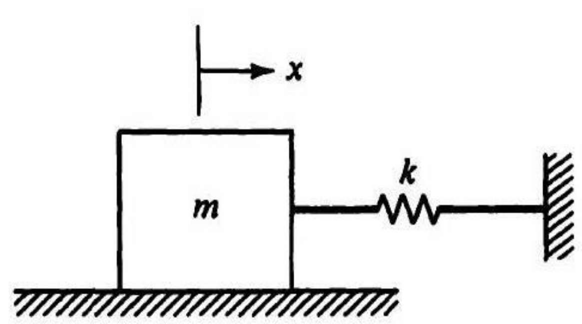


Fig. 4.1. A mass-spring system.

and we find that the Hamiltonian function is equal to the total energy, namely,

$$H = T + V = \frac{p^2}{2m} + \frac{1}{2}kx^2 \quad \text{........... (3)}$$

Since we are considering a conservative system, we can use directly the modified Hamilton-Jacobi equation which, in this case, is

$$\frac{1}{2m}\left(\frac{\partial W}{\partial x}\right)^2 + \frac{1}{2}kx^2 = \alpha \quad \text{............ (4)}$$

where α is the energy constant. Hence we obtain $\frac{\partial W}{\partial x} = \sqrt{2m\left(\alpha - \frac{1}{2}kx^2\right)}$..........(5)

or $W(x,\alpha) = m\omega \int_{x_0}^{x} \sqrt{a^2 - \xi^2}\, d\xi$ ....(6)

where $a = \sqrt{2\alpha/m\omega^2},\ \omega = \sqrt{k/m}$

In general, the lower limit x_0 of the integral is chosen to be either (1) a convenient absolute constant (not a function of the α 's), or (2) a simple zero of $f(\xi)$, where $\sqrt{f(\xi)}$ is the integrand.

This choice is made in order to simplify differentiation under the integral sign and will, of course, be reflected in the meaning attached to the various β 's.

Differentiating with respect to α, we obtain

$$t-\beta=\frac{1}{\omega}\int_{x_0}^{x}\frac{d\xi}{\sqrt{a^2-\xi^2}}=\frac{1}{\omega}\left[\cos^{-1}\frac{x_0}{a}-\cos^{-1}\frac{x}{a}\right] \quad \ldots\ldots\ldots\ldots (7)$$

which yields $x=\sqrt{\frac{2\alpha}{m\omega^2}}\cos[\omega(t-t_0)-\phi]$

where $\cos\phi=\frac{x_0}{a}=\sqrt{\frac{m\omega^2}{2\alpha}}x_0$

and $\beta=t_0$. If we write the total energy α in terms of the initial conditions $x(t_0)=x_0$ and $\dot{x}(t_0)=v_0$, we have $\alpha=\frac{1}{2}mv_0^2+\frac{1}{2}kx_0^2=\frac{m\omega^2}{2}\left(x_0^2+\frac{v_0^2}{\omega^2}\right)$ ………………..(8)

Also $\sin\phi=\frac{1}{a}\sqrt{a^2-x_0^2}=\frac{v_0}{a\omega}$

$$x=x_0\cos\omega(t-t_0)+\frac{v_0}{\omega}\sin\omega(t-t_0) \quad \ldots\ldots\ldots\ldots (9)$$

which is identical with the result obtained by the direct solution of the ordinary differential equation describing the mass-spring system. The amplitude of the oscillation in x is

$$a=\sqrt{x_0^2+\frac{v_0^2}{\omega^2}}$$

In evaluating an integral involving $\sqrt{f(\xi)}$, a question arises concerning which sign is to be chosen for the square root.

It frequently occurs that the variable of integration ξ oscillates between two zeros of $f(\xi)$, indicating a librational motion.

For example, that we must change the sign of $\sqrt{f(\xi)}$ at each turning point in the libration, that is, at the point where the direction of motion in ξ reverses. In this example, $\sqrt{f(\xi)}$ is positive for positive $d\xi$ and is negative for negative $d\xi$. Notice that the turning points occur at zeros of

$f(\xi)$ and, since the integrand may become infinite at these points, it can result in an improper integral. For the usual case of simple zeros, however, this integral converges, indicating a finite period for the librational motion.

In this example we have found a solution for the motion of the mass spring system without the necessity of evaluating the integral of pervious example which leads to an explicit expression for $W(x, \alpha)$. We find that

$$S = -\alpha t + \frac{m\omega}{2}\left(x\sqrt{a^2 - x^2} - a^2\cos^{-1}\frac{x}{a} - x_0\sqrt{a^2 - x_0^2} + a^2\cos^{-1}\frac{x_0}{a}\right) \quad \ldots\ldots\ldots\ldots (10)$$

Here, for the sake of simplicity, we have expressed W as a function of a, which we recall is equal to $\sqrt{2\alpha/m\omega^2}$.

It is a straightforward process to check that this expression for the principal function obeys the Hamilton-Jacobi equation. Furthermore, it leads to the correct solution for the motion of the system upon the application 0°. Thus we obtain

$$-\beta = \frac{\partial S}{\partial \alpha} = -t + \frac{\partial W}{\partial a}\frac{da}{d\alpha} = -t + \frac{1}{\omega}\left(\cos^{-1}\frac{x_0}{a} - \cos^{-1}\frac{x}{a}\right) \quad \ldots\ldots\ldots\ldots (11)$$

(or) $x = a\cos[\omega(t - t_0) - \phi]$ ....(12)

where we have substituted t_0 for β. This result is identical with that found previously in Equation.

Finally, let us calculate the principal function $S(x, x_0, t, t_0)$ by evaluating the canonical integral equation. We have

$$S = \int_{t_0}^{t} (T - V)dt = \frac{m}{2}\int_{t_0}^{t} (\dot{x}^2 - \omega^2 x^2)dt \quad \ldots\ldots\ldots\ldots (13)$$

Performing the integration and simplifying, we obtain

$$S = -\frac{m\omega}{2}\sin\omega(t - t_0)\left[\left(x_0^2 - \frac{v_0^2}{\omega^2}\right)\cos\omega(t - t_0) + \frac{2x_0 v_0}{\omega}\sin\omega(t - t_0)\right] \ldots\ldots\ldots (14)$$

In order to express S in terms of the desired quantities, we note from equation that

$$\frac{v_0}{\omega} = \frac{x - x_0\cos\omega(t - t_0)}{\sin\omega(t - t_0)} \quad \ldots\ldots\ldots\ldots (15)$$

Then, substituting equation (6) into equation (5), we obtain

$$S(x, x_0, t, t_0) = \frac{1}{2}m\omega(x^2 + x_0^2)\cot\omega(t - t_0) - m\omega x x_0\csc\omega(t - t_0) \quad \ldots\ldots\ldots\ldots (16)$$

By comparing this result with the principal function obtained previously in equation (1), we see that there is a considerable difference in form. In particular, this S function has no linear term in t even though we are analyzing a conservative system. Nevertheless, it represents another complete solution of the Hamilton-Jacobi equation and, in accordance with Jacobi's theorem, it allows one to find the motion of the system by a process of differentiations and algebraic manipulations.

To illustrate this point, let us consider that x_0 assumes the role of α, that is, x_0 is an arbitrary constant which in this case describes the position of the system at a preassigned time

t_0. $-\beta = \frac{\partial S}{\partial x_0} = m\omega x_0 \cot\omega(t - t_0) - m\omega x \csc\omega(t - t_0)$ ………….. (17)

Solving for x, we find that

$x = x_0 \cos\omega(t - t_0) + \frac{\beta}{m\omega} \sin\omega(t - t_0)$ ………… (18)

In accordance with the theory, β is equal to the initial momentum mv_0. Hence equation (9) is identical to the previous result given in equation.

Example 2:

Let us use the Hamilton-Jacobi method to analyze the Kepler problem.

Solution:

Suppose a particle of unit mass is attracted by an inverse-square gravitational

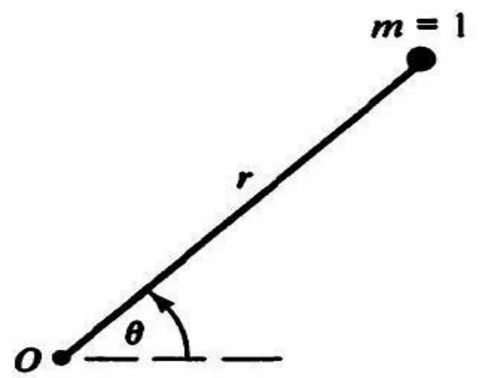


Fig. 4-2. The Kepler problem.

force to a fixed point O (Fig. 4.2). The position of the particle is given in terms of the polar coordinates (r, θ) measured in the plane of the orbit.

The kinetic and potential energies are

$T = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2)$, $V = -\frac{\mu}{r}$ …………….. (1)

where μ is the gravitational coefficient. The Lagrangian function is

$L = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{\mu}{r}$ ………….. (2)

and we find that the generalized momenta are given by

$p_r = \frac{\partial L}{\partial \dot{r}} = \dot{r},\ p_\theta = \frac{\partial L}{\partial \dot{\theta}} = r^2\dot{\theta}$ .... (3)

We are considering a natural system, so the Hamiltonian function is equal to the total energy, that is, $H = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - \frac{\mu}{r} = \alpha_t$ ....(4)

Here we use α_t to represent the constant value of the total energy. The coordinate θ does not appear in H and therefore it is ignorable, implying that the conjugate momentum p_θ has a constant value which we shall designate by α_θ. Then, in accordance with Eq. (5-81), we see that the principal function can be written in the form

$S = -\alpha_t t + \alpha_\theta \theta + W'(r, \alpha_t, \alpha_\theta)$ .... (5)

The modified Hamilton-Jacobi equation is $\frac{1}{2}\left(\frac{\partial W'}{\partial r}\right)^2 + \frac{\alpha_\theta^2}{2r^2} - \frac{\mu}{r} = \alpha_t$ ....(6)

and we obtain $\frac{\partial W'}{\partial r} = \sqrt{2\alpha_t + \frac{2\mu}{r} - \frac{\alpha_\theta^2}{r^2}}$

which results in $W' = \int_{r_0}^{r} \sqrt{2\alpha_t + \frac{2\mu}{r} - \frac{\alpha_\theta^2}{r^2}}\, dr$

where r_0 is the value of the radial distance r at the initial time t_0. Now let us differentiate under the integral sign with respect to α_t.

$t - t_0 = \frac{\partial W'}{\partial \alpha_t} = \int_{r_0}^{r} \frac{dr}{\sqrt{2\alpha_t + \frac{2\mu}{r} - \frac{\alpha_\theta^2}{r^2}}}$ ....(7)

Note that the square root is equal to $\dot{r}$, as may be seen from the energy equation (4)

$\theta - \theta_0 = -\frac{\partial W'}{\partial \alpha_\theta} = \int_{r_0}^{r} \frac{\alpha_\theta dr}{r\sqrt{2\alpha_t r^2 + 2\mu r - \alpha_\theta^2}}$ ....(8)

Comparing these last two integrals, we find that the first gives t as a function of r, whereas the second gives θ as a function of r, that is, it gives the shape of the orbit. It is convenient to measure θ from the position of minimum r. Then $\theta_0 = 0$ for the case where we choose $r_0 = r_{\min}$,

$\theta = \cos^{-1}\left(\frac{\alpha_\theta^2 - \mu r}{r\sqrt{\mu^2 + 2\alpha_t \alpha_\theta^2}}\right)$ .... (9)

Finally, solving for r, we obtain

$$r = \frac{\alpha_\theta^2/\mu}{1 + \sqrt{1 + 2\alpha_t \alpha_\theta^2/\mu^2} \cos\theta}$$

which we recognize as the equation of a conic section having an eccentricity

$e = \sqrt{1 + 2\alpha_1\alpha_\theta^2/\mu^2}$

4.3. SEPARABILITY:

The idea of separability is associated with the solution of partial differential equations by a reduction to quadratures, that is, by expressing the solution in terms of integrals, each involving only one variable. In the context of the Hamilton-Jacobi partial differential equation, the possibility of obtaining a separation of variables depends partly upon the nature of the physical system and partly upon the coordinates used in its mathematical representation. Quite naturally, we would like to know the conditions under which such a separation is possible. Unfortunately, the answer to the basic question concerning what is the most general separable system is not known. Some progress can be made, however, if we restrict the investigation to a certain class of systems. In particular, we shall consider in the following discussion only orthogonal systems, that is, conservative holonomic systems whose kinetic energy function contains only squared terms in the $\boldsymbol{q}$'s (or p 's), and no product terms in these variables. In other words, there are no inertial coupling terms.

We shall assume that the term separability implies that a characteristic function for the system can be found which has the form $W = \sum_{i=1}^{n} W_l(q_i)$ ....(1)

that is, it consists of the sum of $\boldsymbol{n}$ functions where each function W_1 contains only one of the q 's. Furthermore, we shall assume that W is a complete integral of the modified Hamilton-Jacobi equation and thus contains n nonadditive constants (including the energy constant), usually designated by α 's. A particularly simple example of a separable system occurs in the case where all but one of the coordinates are ignorable.

Liouville's System:

Let us define a Liouville system to be an orthogonal system which has kinetic and potential energies of the forms $T = \frac{1}{2}\left(\sum_{n=1}^{n} f_i(q_t)\right)\left(\sum_{n=1}^{n} \frac{q_q^2}{R_t(q_t)}\right) = \frac{R_1p_1^2+\cdots+R_np_n^2}{2(f_1+\cdots+f_n)}$ ....(2)

$V = \frac{v_1(q_1)+\cdots+v_n(q_n)}{f_1(q_1)+\cdots+f_n(q_n)}$ .... (3)

where f_t, R_t, and v_t are each functions of q_t, and we note that R_t is identical with M_i^{-1}. Also, we assume that $\sum_i f_i(q_t) > 0$ and $R_t(q_t) > 0$.

These Liouville conditions are sufficient to ensure the separability of the given system, and therefore a reduction to quadratures is possible. As a proof, we can show that a complete solution $W(q)$ of the modified Hamilton Jacobi equation exists, and this solution has the separable form given in Eq. (1).

The modified Hamilton-Jacobi equation for this system can be written in the form

$$\sum_{i=1}^{n} \left[\frac{1}{2} R_i \left(\frac{\partial W}{\partial q_t}\right)^2 + v_t\right] = h \sum_{i=1}^{n} f_i \quad \ldots\ldots\ldots\ldots (4)$$

Now let us substitute for W from equation (1). We shall find that each function $W_l(q_t)$ can be obtained in integral form and a complete solution results. First, let us group the terms in each coordinate $q_i (i = 1,2, \ldots, n)$ and use $\alpha_1, \alpha_2, \ldots, \alpha_n$ as separation constants. Upon setting each group of terms equal to the corresponding α_l, we have

$$\frac{1}{2} R_t \left(\frac{\partial W_t}{\partial q_t}\right)^2 + v_t - hf_t = \alpha_t \; (i = 1,2, \ldots, n) \quad \ldots\ldots\ldots\ldots (5)$$

Where $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 0 \quad \ldots\ldots\ldots\ldots\ldots(6)$

Eq. (5) can be integrated and the resulting W_i 's added in accordance with Eq. (1) to obtain

$$W = \sum_{i=1}^{n} \int \frac{1}{R_i} \sqrt{\phi_i(q_t)} dq_t \quad \ldots\ldots\ldots\ldots (7)$$

Where $\phi_t(q_t) = 2R_t[hf_t(q_t) - v_t(q_i) + \alpha_t] \; (i = 1,2, \ldots, n)$

This solution actually contains the ($n + 1$) constants $\alpha_1, \alpha_2, \ldots, \alpha_n, h$, but equation (6) represents one relation among the α 's. Therefore, one α_t can be eliminated, leaving the required $\boldsymbol{n}$ independent constants. Hence we see that the expression for W given in equation (7) is a complete solution of the modified Hamilton-Jacobi equation, thereby confirming that the Liouville conditions are sufficient for the separability of an orthogonal system.

We turn next to the solution for the motion of the system, assuming that we have found the characteristic function $W(q)$. First, let us arbitrarily eliminate α_n by regarding it as a function of the other α 's, that is, $\alpha_n = -(\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1})$

Then we have $\frac{\partial W}{\partial \alpha_t} = \frac{\partial W_t}{\partial \alpha_t} + \frac{\partial W_n}{\partial \alpha_n}\frac{\partial \alpha_n}{\partial \alpha_t} = \frac{\partial W_t}{\partial \alpha_t} - \frac{\partial W_n}{\partial \alpha_n} \; (i = 1,2, \ldots, n-1)$

$$\frac{\partial W}{\partial \alpha_t} = \int \frac{dq_i}{\sqrt{\phi_i(q_t)}} - \int \frac{dq_n}{\sqrt{\phi_n(q_n)}} = -\beta_i \; (i = 1,2, \ldots, n-1) \ldots\ldots\ldots\ldots(8)$$

Also, we find that $\frac{\partial W}{\partial h} = \sum_{i=1}^{n} \int \frac{f_i dq_i}{\sqrt{\phi_i(q_i)}} = t - \beta_n \quad \ldots\ldots\ldots\ldots\ldots(9)$

The solution to the Lagrange problem is given by equations. (8) and (9) and presents the path of the system in extended configuration space. The path in phase space is found by using the additional equations $p_t = \frac{\partial W}{\partial q_t} = \frac{1}{R_t}\sqrt{\phi_t(q_i)}\ (i = 1,2,\dots,n)$

Since β_i in equation(8) is a constant along any actual path of the system, we see that the increments in the values of any two of the given integrals must be equal for any interval of time. Hence we obtain $\frac{dq_1}{\sqrt{\phi_1(q_1)}} = \frac{dq_2}{\sqrt{\phi_2(q_2)}} = \cdots = \frac{dq_n}{\sqrt{\phi_n(q_n)}} = d\tau$

Theorem 1: (Stakel's & Theorem)

State and prove Stakel's & Theorem:

An orthogonal System is separable if and only if the following two conditions are satisfied

(i.e.,) A non-singular $n \times n$ matrix.$[\Phi_{ij}(q_i)]$ and the column matrix $[\psi_i(q_i)]$ exists.

Such that,(1) $C^{\top}\Phi = [1,0,0,\cdots 0]$

(2) $C^T\psi = v$, where $v(= q_1, q_2 - \cdots q_n)$ is P, E where C is the column matrix of n c's.

Proof:

Necessary part:

Let the given orthogonal system be separable.

⇒ The characteristic function $w(q, \alpha)$.

$$w(q,\alpha) = \sum_i w_i(q_i, \alpha_1, \dots \alpha_n)$$

This character function is a complete integral of modified Hamilton Jacobi equation.

(i.e.,), $\frac{1}{2}\sum_{i=1}^{n} c_i \left(\frac{\partial w_i}{\partial q_i}\right)^2 + v = \alpha_i$ ....(1) $\alpha_i =$ Total Energy

For the orthogonal system,

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i(\dot{q}_i)^2$$

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i p_i^2$$

Because of the definition of separable, ω_i is ' 0 ' function of generalized co-ordinate q_i only.

function α possibly the constants $\alpha_1, \alpha_2 \cdots \alpha_n$ is $\frac{\partial w_i}{\partial q_i}$.

Hence we can rewrite the above equation in the form, The must integral form involving the single coordinates $q_i{}'s\ \left(\frac{\partial w_i}{\partial q_i}\right)^2 = -2\psi_i(q_i) + 2\sum_{j=1}^{n} \Phi_{ij}(q_i)\alpha_j$ ………………(2)

Where the constants are chosen for convenient

$\frac{1}{2}\sum_{i=1}^{n} c_i\big(-2\psi_i(q_i) + 2\sum_{j=1}^{n}, \Phi_{ij}(a_i)\alpha_j + V = \alpha_i$

Now sub this in (1), we can get an equation in matrix form

$-c^{\top}\psi + c^{\top}I_\alpha + v = \alpha_i$ ………… (3)

Comparing the term containing α. we find the

$C^{\top}\Phi = [1,0,0, \dots - 0]$ ………… (4)

which is called first Stakel's condition's similarly equation(1) which does not involve α' 's must Some to zero. $C^{\top} \cdot \psi = V$ … … … … (5)

This is called second Stakel's condition.

Sufficient part:

Assume that a given orthogonal system Satisfies the Stakel's conditions [both].

For convenient,

column Matrix a is denoted as $a_i = \left(\frac{\partial w}{\partial q_i}\right)^2$, … … … (6) $i = 1$ to n

$$\text{(i.e.,) } a = \begin{pmatrix} \left(\frac{\partial \omega}{\partial q_1}\right)^2 \\ \left(\frac{\partial \omega}{\partial q_2}\right)^2 \\ \vdots \\ \left(\frac{\partial \omega}{\partial q_n}\right)^2 \end{pmatrix}$$

We can write the modified Hamilton Jacobi equation,

$$\frac{1}{2}\sum c_i \left(\frac{\partial w_i}{\partial q_i}\right)^2 + v = \alpha_1 \text{ (or)}$$

$\frac{1}{2}\sum c_i \left(\frac{\partial w_i}{\partial q_i}\right)^2 + v = \alpha_1$ is a matrix Equation.

$$\frac{1}{2}c^{\top}a + v = \alpha_1 \dots \dots (7) \text{ [equation (5)]}$$

$$\Rightarrow \frac{1}{2}c^{\top}a + c^{\top}\Psi = [1,0,\cdots 0]\alpha \quad \dots \dots \dots (8) \text{ [by } 2^{nd} \text{ stakel's condition]}$$

⇒ Also first Stakel's condition,

(4) $\Rightarrow C^{\top}\Phi = [1,0,\dots 0]$

$\Rightarrow C^{\top} = [1,0,\dots 0]\Phi^{-1}$ .... (9)

Sub in above equation,

$$\frac{1}{2}[1,0,\cdots - 0]\Phi^{-1}a + [1,0,\dots - 0\Phi^{-1}\psi = [1,0,- - 0]\alpha.$$

$$[1,0 \dots \dots .0]\left(\frac{1}{2}\Phi^{-1}a + \Phi^{-1}\psi\right) = [1,0,\dots \dots 0]\alpha$$

$$\frac{1}{2}\Phi^{-1}a + \Phi^{-1}\psi = \alpha.$$

Which has a Solution $a = -2\psi + 2\,\Phi\alpha$ .... (10)

Further Simplification,

$$\left(\frac{\partial w_i}{\partial q_i}\right)^2 = -2\psi(q_i) + 2\sum_j \Phi_{ij}(q_i)\alpha_j$$

Hence the system is separable.

Example 1:

Let us consider the Kepler problem once again, this time inquiring into its separability. Suppose we use spherical coordinates to specify the position of the particle of unit mass (Fig. 4-3) which is attracted to the fixed point O by an inverse-square gravitational force.

The kinetic and potential energies are

$T = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta\right)$ and $V = -\frac{\mu}{r}$ ....(1)

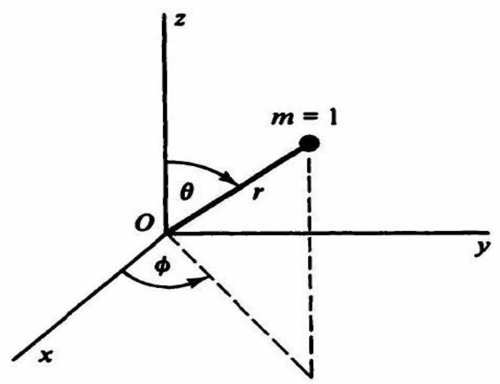


Fig. 4-3. The Kepler problem, using spherical coordinates.

where p_r, p_θ, p_ϕ are the generalized momenta. Then, since this is an orthogonal system, we know that the Hamiltonian $H = T + V$ is constant, and the modified Hamilton-Jacobi equation is $\frac{1}{2}\left(\frac{\partial W_r}{\partial r}\right)^2 + \frac{1}{2r^2}\left(\frac{\partial W_\theta}{\partial \theta}\right)^2 + \frac{1}{2r^2\sin^2\theta}\left(\frac{\partial W_\phi}{\partial \phi}\right)^2 - \frac{\mu}{r} = \alpha_t$ ....(1)

where $W = W_r(r) + W_\theta(\theta) + W_\phi(\phi)$ …………(2)

In other words, we seek a characteristic function which is a complete solution of the modified Hamilton-Jacobi equation and which has the separable form of equation. (2).

At this point we note that ϕ is missing from T and V, and is therefore an ignorable coordinate.

Hence $p_\phi = \frac{\partial W_\phi}{\partial \phi} = \alpha_\phi$

and it follows that $W_\phi(\phi) = \alpha_\phi \phi$

So far, we have obtained two of the required three α 's. The third is found by obtaining a separation of variables through first multiplying equation (1) by $2r^2$, yielding

$$r^2\left(\frac{\partial W_r}{\partial r}\right)^2 - 2r^2\left(\frac{\mu}{r} + \alpha_t\right) + \left(\frac{\partial W_\theta}{\partial \theta}\right)^2 + \frac{\alpha_\phi^2}{\sin^2\theta} = 0 \quad \text{……… (3)}$$

Here the first two terms are functions of r only, and the last two terms are functions of $\boldsymbol{\theta}$ only. Hence, they are each equal to a separation constant,

that is, we can take $\left(\frac{\partial W_\theta}{\partial \theta}\right)^2 + \frac{\alpha_\phi^2}{\sin^2\theta} = \alpha_\theta^2$ …………(4)

And $r^2\left(\frac{\partial W_r}{\partial r}\right)^2 - 2r^2\left(\frac{\mu}{r} + \alpha_t\right) = -\alpha_\theta^2$ …………(5)

The separation constant is chosen to be α_θ^2 (rather than α_θ) as a matter of convenience. This choice permits α_θ to have the dimensions of angular momentum, but does not influence the validity of the Stackel conditions.

We are now assured that the system is separable because

we have $\frac{\partial W_r}{\partial r} = \sqrt{2\left(\frac{\mu}{r} + \alpha_t\right) - \frac{\alpha_\theta^2}{r^2}}$

And $\frac{\partial W_\theta}{\partial \theta} = \sqrt{\alpha_\theta^2 - \frac{\alpha_\theta^2}{\sin^2\theta}}$

which are immediately integrable. W_ϕ was found previously.

Now let us check to see that the Stackel conditions are met. we see that $c = \left\{1, \frac{1}{r^2}, \frac{1}{r^2 \sin^2\theta}\right\}$

we obtain $\Phi = \begin{bmatrix} 1 & -\frac{\alpha_\theta}{r^2} & 0 \\ 0 & \alpha_\theta & -\frac{\alpha_\phi}{\sin^2\theta} \\ 0 & 0 & \alpha_\phi \end{bmatrix}$

We immediately see that if we choose $\Psi = \left\{-\frac{\mu}{r}, 0, 0\right\}$

It is interesting to note that this separable system does not meet the criteria of a Liouville system for $n = 3$, thereby illustrating that the Liouville conditions are not necessary conditions. By comparing with the kinetic energy expression of equation (1), we see that the Liouville conditions require that

$$\frac{R_\phi(\phi)}{R_r(r)} = \frac{1}{r^2 \sin^2 \theta}$$

which is impossible.

Another approach to this problem is to take advantage immediately of the fact that ϕ is an ignorable coordinate, thereby reducing the number of

degrees of freedom from three to two. We can write

$$H = \frac{1}{2}p_r^2 + \frac{1}{2r^2}p_\theta^2 + \frac{\alpha_\phi^2}{2r^2 \sin^2 \theta} - \frac{\mu}{r} = \alpha_t \quad \text{............ (6)}$$

and then regard this as an orthogonal system with kinetic and potential energies given by

$$T' = \frac{1}{2}p_r^2 + \frac{1}{2r^2}p_\theta^2 \quad \text{............. (7)}$$

And $V' = \frac{\alpha_\phi^2}{2r^2 \sin^2 \theta} - \frac{\mu}{r}$ ....(8)

The characteristic function is of the form $W = W_r(r) + W_\theta(\theta)$

and the modified Hamilton-Jacobi equation is

$$\frac{1}{2}\left(\frac{\partial W_r}{\partial r}\right)^2 + \frac{1}{2r^2}\left(\frac{\partial W_\theta}{\partial \theta}\right)^2 + \frac{\alpha_\phi^2}{2r^2 \sin^2 \theta} - \frac{\mu}{r} = \alpha_t \quad \text{............ (9)}$$

Once again, the system proves to be separable and Equations (4) apply.

A check of the Stackel conditions shows that c $= \left\{1, \frac{1}{r^2}\right\}$

$$\Phi = \begin{bmatrix} 1 & -\frac{\alpha_\theta}{r^2} \\ 0 & \alpha_\theta \end{bmatrix}$$

And $\Psi = \left\{-\frac{\mu}{r}, \frac{\alpha_\phi^2}{2\sin^2 \theta}\right\}$

For this case in which $\boldsymbol{n} = 2$, we know that the Liouville and Stäckel criteria give identical results. A comparison of the expressions for T' and V' with equations reveals that

$$f_r(r) = r^2, \qquad f_\theta(\theta) = 0$$
$$R_r(r) = r^2, \qquad R_\theta(\theta) = 1$$
$$v_r(r) = -\mu r, \quad v_\theta(\theta) = \frac{\alpha_\phi^2}{2\sin^2 \theta}$$

thereby confirming that the Liouville's conditions apply. Notice that equivalent expressions for T' and V' would have been obtained if the Routhian procedure had been used with respect to the ignorable coordinate ϕ, rather than the Hamiltonian approach employed here.

UNIT- V

Canonical Transformations: Differential forms and generating functions – Special Transformations– Lagrange and Poisson brackets.

Chapter 5: Sections 5.1-5.3

5.1. Canonical Transformations:

Definition:

1. A transformation from (q, p) to (Q, p) which preserves the Canonical form of equation of motion is known as a canonical transformation

2. Show that the canonical transformation from a group.

Proof:

We know that,

$$\sum_{i=1}^{n} P_i \dot{q}_i - H = \sum_{i=1}^{n} P_i \dot{Q}_i - k + \frac{dF}{dt}$$

$$\Rightarrow \sum_{i=1}^{n} P_i dq_i - Hdt - \sum_{i=1}^{n} P_i dQ_i + kdt = dF \dots\dots\dots..(1) \; F = F,(q, Q, t)$$

where $p_i = \frac{\partial F}{\partial q_i}\}$

$$\left.\begin{array}{l} P_i = -\dfrac{\partial F}{\partial Q_i} \\ k = H + \dfrac{\partial F}{\partial t} \end{array}\right\} \quad \dots\dots\dots\dots\dots(2)$$

The symmetry of equation (1) & (2), shows that the inverse of given canonical function is a Canonical function.

The canonical function is derived by negative of $F(q, Q, t)$.

Note that, the sum of two exact differentials. expressed in terms of old variable (q, Q, t) is also exact.

Two canonical transformation succession result in an canonical transformation, clearly we can also have an identity transformation $F = \sum_{i=1}^{n} q_i \dot{p}_i$, which is also canonical.

Hence by given value of n the cannonical transformation form a group.

Theorem 1:

Derive the condition for transformation to be Canonical.

Proof:

consider (q,p) into (Q,p) with H & K has the Hamiltonian.

$$\dot{Q}=\frac{dQ}{dt}=\frac{\partial Q}{\partial q}\dot{q}+\frac{\partial Q}{\partial p}\dot{p}$$

$$\dot{Q}=\frac{\partial Q}{\partial q}\cdot\frac{\partial H}{\partial p}-\frac{\partial Q}{\partial p}\cdot\frac{\partial H}{\partial q}$$ [by Hamilton Canonical equation]

Also $\frac{dH}{dp}=\frac{\partial H}{\partial q}\cdot\frac{dq}{dp}+\frac{\partial H}{\partial p}\cdot\frac{dp}{dp}\,[\because H(p,q)P(Q,p),q(Q,p)]$

But $\dot{Q}=\frac{\partial H}{\partial p}$

(i.e.,) $\frac{\partial Q}{\partial q}\cdot\frac{\partial H}{\partial p}-\frac{\partial Q}{\partial p}\cdot\frac{\partial H}{\partial q}=\frac{\partial H}{\partial q}\cdot\frac{\partial q}{\partial p}+\frac{\partial H}{\partial p}\cdot\frac{\partial p}{\partial p}.$

Equate co-eff of $\frac{\partial H}{\partial P}:\frac{\partial Q}{\partial q}=\frac{\partial P}{\partial P}\Big\}$ Equate co-eff of $\frac{\partial H}{\partial q}:\frac{\partial Q}{\partial p}=-\frac{\partial q}{\partial p}\Big\}$ ....(1)

Also $\dot{p}=\frac{-\partial H}{\partial Q}.$

$$\frac{\partial p}{\partial q}\cdot\dot{q}+\frac{\partial p}{\partial p}\cdot\dot{p}=-\left[\frac{\partial H}{\partial p}\cdot\frac{\partial p}{\partial Q}+\frac{\partial H}{\partial q}\cdot\frac{\partial q}{\partial Q}\right]$$

$$\frac{\partial p}{\partial q}\cdot\frac{\partial H}{\partial p}-\frac{\partial p}{\partial p}\cdot\frac{\partial H}{\partial q}=-\frac{\partial p}{\partial Q}\cdot\frac{\partial H}{\partial p}-\frac{\partial q}{\partial Q}\cdot\frac{\partial H}{\partial q}.$$

Equate like term co-eff, $\left.\begin{matrix}\frac{\partial p}{\partial q}=-\frac{\partial p}{\partial Q}\\ \frac{\partial p}{\partial p}=+\frac{\partial q}{\partial Q}\end{matrix}\right\}$ ....(2)

By Jacobian,

$$J\left(\frac{Q,p}{q,p}\right)=\begin{vmatrix}\frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p}\\ \frac{\partial p}{\partial q} & \frac{\partial p}{\partial p}\end{vmatrix}$$

$$=\frac{\partial Q}{\partial q}\cdot\frac{\partial p}{\partial p}-\frac{\partial Q}{\partial p}\cdot\frac{\partial p}{\partial q}$$

$$=\left(\frac{\partial p}{\partial p}\right)\left(\frac{\partial q}{\partial Q}\right)-\left(-\frac{\partial q}{\partial p}\right)\left(\frac{-\partial p}{\partial Q}\right)$$

$$=\frac{\partial q}{\partial Q}\cdot\frac{\partial p}{\partial p}-\frac{\partial q}{\partial p}\cdot\frac{\partial p}{\partial Q}.$$

(i.e.,) $\frac{\partial(Q,p)}{\partial(p,q)}=\frac{\partial(q,p)}{\partial(Q,p)}.$

(or) $\frac{\partial(Q,p)}{\partial(p,q)}=\frac{1}{\frac{\partial(Q,p)}{\partial Q(q,p)}}$

(or) $\left[\frac{\partial(Q,p)}{\partial(p,q)}\right]^2=1.$ (or) $\left[\frac{\partial(Q,p)}{\partial(p,q)}\right]=1.$

This is the condition for $(q, p) \to (Q, p)$ to be Canonical Transformation.

Define orthogonal transformation:

Consider $F_2 = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} p_i q_j$

where a' s are constant meeting the orthogonality condition.

$$\sum_{i=1}^{n} a_{ij} a_{ik} = \delta_{jk} \to \text{[kronical delta]}$$

(i.e.,) $\delta_{jk} = \begin{cases} 0 & \text{if } j = k \\ 1 & \text{if } j \neq k \end{cases}$

The transformation equation are $P_j = \frac{\partial F_2}{\partial q_j} = \sum_{i=1}^{n} a_{ij} P_i$

$$Q_i = \frac{\partial F_2}{\partial p_i} = \sum_{j=1}^{n} a_{ij} q_j$$

Now, $\sum_{k=1}^{n} P_j a_{jk} = \sum_{k=1}^{n}\sum_{i=1}^{n} a_{ij} a_{jk} P_i$.

$$= \sum_{k=1}^{n} \delta_{jk} P_i$$

$$= P_i, \text{ where } j = k$$

$$\Rightarrow P_i = -\sum_{j=1}^{n} P_j a_{ij}$$

Also, $Q_i = \frac{\partial F_2}{\partial p_i} = \sum_{i=1}^{n} a_{ij} q_j$

Where $a_{ij} = \delta_{ij}$

$\Rightarrow$ The identily transformation

$$P_i = P_i$$
$$Q_i = q_i$$

Definition: Homogeneous Canonical Transformation:

Let $(q, p) \to (Q, p)$ be the canonical transformation of the old and new set of canonical co-ordinates.

We know that $\sum_{i=1}^{n} P_i \dot{q}_i - H = \sum_{i=1}^{n} P_i \dot{Q}_i - k + \frac{dF}{dt}$

(i.e.,) $\sum_{i=1}^{n} p_i \delta q_i - H(q, p, t) - \sum_{i=1}^{n} P_i \delta Q_i + K(Q, P, t) = \delta_k$

Suppose " t " is invariant (constant)

$$\delta t = 0 \text{ also } \delta F = 0$$

$$\sum_{i=1}^{n} P_i \delta q_i - \sum_{i=1}^{n} P_i \delta Q_i = 0$$

This transformation is called Homogenous canonical transformation.

Result -1

This is also knowns Mathieu transformation (or) Contact transformation.

Result-2

In general, $k = H + \sum_{i=1}^{n} P_i \frac{\partial Q_i}{\partial t}$.

for scleronomic system.

Define Bilinear covariant:

consider the Pfaffian differential form,

$\Omega = \sum_{i=1}^{n\infty} X_i(x) dx_i$

Let $\theta = \sum_{j=1}^{m} x_j \delta x_j$ then $\delta w - \delta\theta = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} dx_i dx_j$

where $c_{ij} = \frac{\partial x_i}{\partial x_j} - \frac{\partial x_j}{\partial x_i}$ is called "Bilinear covariant"

Example 1:

Prove that Bilinear covariant is invariant with respect to canonical transformation

Proof:

Let (q,p) be the set of $2n$ cannonical co-ordinates which are transformed into (q,p) by canonical transformation

∴ consider the differentiation form

$$\sum_{j=1}^{n} p_i dq_i - \sum_{i=1}^{n} P_i dQ_i = d\psi(q,p)$$

where $d\psi$ is an exact differential and we consider time as a parameter. Then we have

$\delta(\sum_{i=1}^{n} p_i dq_i - \sum_{i=1}^{n} p_i dQ_i) = \delta d\psi(q,p)$

(ie) $\sum_{i=1}^{n} \delta p_i dq_i - \sum_{i=1}^{n} \delta p_i dQ_i = \delta(d\psi(q,p))$

Similarly, $d(\sum_{i=1}^{n} p_i \delta q_i - \sum_{i=1}^{n} p_i \delta Q_i) = d\big(\delta\psi(q,p)\big)$ ... ...(1)

(i.e.,) $(\sum_{i=1}^{n} dp_i \cdot \delta q_i) - \sum_{i=1}^{n} dp_i \delta Q_i = d(\delta\psi(q,p))$ ....(2)

(1) - (2) ⇒ $\sum_{i=1}^{n} (\delta p_i dq_i - dp_i \delta q_i) - (\sum_{i=1}^{n} \delta p_i dQ_i - \sum_{i=1}^{n} dp_i \delta Q_i)$

$= ds\psi - \delta d\psi$

$= 0$ [by known result]

$$\Rightarrow \sum_{i=1}^{n} (\delta p_i dq_i - \delta q_i \cdot dp_i) = \sum_{i=1}^{n} (\delta p_i dQ_i - \delta Q_i dp_i)$$

⇒ hence Bilinear covariant is invariant.

Example 2:

Show that the transformation $Q = \frac{1}{2}(q^2 + p^2)$, P= $-\tan^{-1}(q/p)$ is canonical and find its generating function.

Proof:

Consider $p\delta q - p\delta Q = p\delta q + \tan^{-1}(q/p)\left[\frac{1}{2}(2q\delta q + 2p\delta p)\right]$

$p\ \delta q = P\delta Q = (p + \delta q\tan^{-1}(q/p)]\delta q + (p\tan^{-1}(q/p)\delta p)]$

The verify -canonical.

To prove that (1) Exact

$$\frac{\partial M}{\partial p} = \frac{\partial N}{\partial q}$$

$$for\ \frac{\partial M}{\partial p} = \frac{\partial}{\partial p}[p + q\tan^{-1}(q/p)]$$

$$= 1 + q \cdot \frac{1}{1 + q^2/p^2}(-q/p^2)$$

$$= 1 - \frac{q^2}{p^2} \cdot \frac{p^2}{p^2 + q^2}$$

$$\therefore \frac{\partial M}{\partial p} = \frac{p^2}{p^2 + q^2}$$

Similarly,

$$\frac{\partial N}{\partial q} = \frac{\partial}{\partial q}[p\tan^{-1}(q/p)]$$

$$= p \cdot \frac{1}{1 + \frac{q^2}{p^2}}\left(\frac{1}{p}\right)$$

$$\frac{\partial N}{\partial q} = \frac{p^2}{p^2 + q^2}$$

∴ (1) is exact

Hence the given transformation is canonical.

To find Ψ :-

Take $\frac{\partial\psi}{\partial q} = p + q\tan^{-1}(q/p)$

$$\frac{\partial\psi}{\partial p} = p\tan^{-1}\cdot(q/p)$$

Integrate partially with respect to P

$$\Psi = \int p\tan^{-1}\left(\frac{q}{p}\right)dp$$

$$= \int \tan^{-1}\left(\frac{q}{p}\right)d\left(\frac{p^2}{2}\right)$$

$$= \left[\frac{p^2}{2}\tan^{-1}(q/p)\right] - \int \frac{p^2}{2}\cdot\frac{p^2}{p^2+q^2}\left(\frac{-q}{p^2}\right)dp$$

$$= \frac{p^2}{2}\tan^{-1}\left(\frac{q}{p}\right) + \frac{q}{2}\int \frac{p^2}{p^2+q^2}dp$$

$$= \frac{p^2}{2}\tan^{-1}\left(\frac{q}{p}\right) + \frac{q}{2}\left[p + \frac{q^2}{q}\tan^{-1}\left(\frac{q}{\mathrm{p}}\right)\right]$$

$$= \frac{p^2}{2}\tan^{-1}(q/p) + \frac{q^2}{2}\tan^{-1}(q/p) + pq/2$$

$$= \frac{pq}{2} + \frac{1}{2}(p^2+q^2)\tan^{-1}(q/p)$$

To find Generation $\phi(q, Q, t)$

Change P into Q and q

$$= \frac{q}{2}\sqrt{2Q-q^2} + \frac{1}{2}[q^2+2Q-q^2]\tan^{-1}\left(\frac{q}{\sqrt{2Q-q^2}}\right)$$

$$= \frac{q}{2}\sqrt{2Q-q^2} + Q\tan^{-1}\left(\frac{q}{\sqrt{2Q-q^2}}\right)$$

$$\phi = \frac{q}{2}\sqrt{2Q-q^2} + Q\tan^{-1}\left(\frac{q}{\sqrt{2Q-q^2}}\right)$$

By known Result:

$$k = H + \frac{\partial\phi}{\partial t}$$

$$\Rightarrow k = H + \frac{1}{2}(q^2+p^2) = Q$$

$$\Rightarrow k = Q$$

The canonical equation of motion in new variable are,

$$Q^* = \frac{\partial k}{\partial p} = 0 \Rightarrow Q = \text{ constant}$$

$$p^* = -\frac{\partial k}{\partial Q} = -1.$$

$\Rightarrow P = -t + c$ [P is decrease in time]

$\Rightarrow \dfrac{\partial K}{\partial Q} = 1.$

$\Rightarrow$ New Hamilton k is constant.

To Find Generalization Function of first kind

$P = \frac{\partial F_1}{\partial q}$ ………… (1)

$P = -\frac{\partial F_1}{\partial Q}$ ………… (2)

$$\frac{\partial F_1}{\partial q} = P = \sqrt{2Q - q^2}$$

Integrate partially with respect to q,

$$F_1 = \int \sqrt{2Q - q^2 dq}$$

By formula,

$$\int \sqrt{a^2 - x^2}dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}(x/a) + c$$

$$\therefore F_1 = \frac{q}{2}\sqrt{2Q - q^2} + \frac{2Q}{2}\sin^{-1}\left(\frac{q}{\sqrt{2Q}}\right) + \phi(Q)$$

By using $P = -\frac{\partial F_1}{\partial Q}$

$$= -\left[\frac{q}{2}\frac{1}{2\sqrt{2Q - q^2}} \cdot 2 + 1\sin^{-1}\left(\frac{q}{2Q}\right) + Q\frac{1}{\frac{\sqrt{1 - q^2}}{2\theta}}[q(-1/2)(2Q)^{-\frac{1}{2}-1}2] + \phi'(Q)\right]$$

$$= -\left[\frac{q}{2}\frac{1}{\sqrt{2Q - q^2}} + \sin^{-1}\left(\frac{q}{\sqrt{2Q}}\right) - \frac{Qq}{\sqrt{2Q - q^2}} \cdot \frac{1}{\sqrt{2Q}} + \frac{1}{(2q)^{1+\frac{1}{2}}} + \phi'(Q)\right.$$

$$P = -\left[\sin^{-1}\left(\frac{q}{2Q}\right) + \phi'(Q)\right]$$

$$= -\sin^{-1}[(-\sin p) - \phi'(Q)]$$

$$P = P - \phi'(Q)$$

This constant may be ignored because transformation equation.

we consider partial derivative.

$\therefore F_1 = q/2\sqrt{2\alpha - q^2} + 2\sin^{-1}(q/\sqrt{2}Q).$

Example 3:

Show that the following Rheonomic transformation

$Q = \sqrt{2q} \cdot e^t \cos P \cdot P = \sqrt{2q} \cdot e^{-t} \sin P$

Proof:

Given transformation, $\left.\begin{array}{l} Q = \sqrt{2q} e^t \cos p \\ P = \sqrt{2q} e^{-t} \sin p \end{array}\right\}$ ……….(1)

To prove that (1) is canonical Transformation

(i) To prove that $\frac{\partial(Q,p)}{\partial(q,p)} = 1.$

$$\Rightarrow \begin{vmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial p}{\partial q} & \frac{\partial p}{\partial p} \end{vmatrix} = \begin{vmatrix} \frac{1}{2\sqrt{2q}} 2e^t \cos p & -\sqrt{2q} e^t \sin p \\ \frac{1}{2\sqrt{2q}} 2e^{-t} \sin p & \sqrt{2q} e^{-t} \cos p \end{vmatrix}$$

$$\frac{\partial(Q,p)}{\partial(q,p)} = \cos^2 p + \sin^2 p = 1$$

∴ (1) is canonical transformation.

Consider,

$$p\delta q - P\delta Q = p\delta q - \sqrt{2q} e^{-t} \sin p \left[\frac{1}{2\sqrt{2q}} \cdot 2 \cdot 1 \cdot e^t \cos p \delta q - \sqrt{2q} e^t \sin p \delta p\right]$$

$$= p\delta q - \sin p \cos p \delta q + 2q \sin^2 p \delta p$$
$$= (p - \sin p \cos p)\delta q + (2q \sin^2 p)\delta p$$

Canonical to $p, \frac{\partial M}{\partial p} = \frac{\partial N}{\partial q}$

To find φ :

$$\frac{\partial \psi}{\partial q} = p - \sin p \cos p$$
$$\frac{\partial \psi}{\partial p} = 2q \sin p$$
$$= 2q\left(\frac{1 - \cos 2p}{2}\right)$$
$$= q(1 - \cos 2p)$$

Integrate partially with respect to

$$\psi = q\left(p - \frac{\sin 2p}{2}\right)$$
$$= pq - \frac{q}{2} \cdot 2 \sin p \cos p$$
$$= pq - q \sin p \cos p$$

Replace p by (Q, q) by Q

$$Q = \sqrt{2q}e^t \cos p$$
$$\frac{Q}{\sqrt{2q}}e^{-t} = \cos p$$
$$p = \cos^{-1}\left(\frac{Q}{\sqrt{2q}}e^{-t}\right)$$

To find ϕ

$$\phi(q,\phi,t) = q\cos^{-1}\left(\frac{Qe^{-t}}{\sqrt{2q}}\right) - q\left(\sqrt{1-\frac{Q^2e^{-2t}}{2q}}\right)\frac{Qe^2}{\sqrt{2}}$$
$$= q\cos^{-1}\left(\frac{Qe^{-t}}{\sqrt{2q}}\right) - q\left(\frac{\sqrt{2q-Q^2e^{-2t}}}{2q}Qe^{-t}\right)$$
$$\phi(q,Q,t) = q\cos^{-1}\left(\frac{Qe^{-t}}{\sqrt{2q}}\right) - \frac{Qe^{-t}}{2}\sqrt{2q-Q^2e^{-2t}}$$

Example 4:

Consider the transformation $Q = \log\frac{\sin p}{q}, P = q\cot p$

Let us obtain the four major types of generating functions associated with this transformation.

Solution:

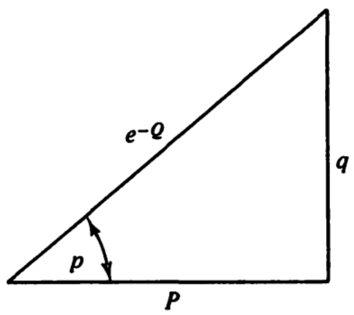


Fig.5.1. A geometrical representation of the transformation

$$p\delta q - p\delta Q = p\delta q - q\cot p\delta\left(\log\frac{\sin p}{q}\right)$$
$$= p\delta q - q\cot p\frac{q}{\sin p}\left[\frac{q\cos p\delta p - \sin p\delta q}{q^2}\right]$$
$$= p\delta q - q\cot p\left(\cot p\delta p - \frac{\delta q}{q}\right)$$
$$= p\delta q + \cot p\delta q - q\cot^2 p\delta p$$
$$= \frac{(p+\cot p)\delta q}{M} - \frac{q\cot^2 p\delta p}{N}.$$
$$\frac{\partial\psi}{\partial q} = p + \cot p.$$

Integrate, $\Psi = pq + q\cot p$

$F_1(q,Q) = \psi(q,Q)$

$Q = \frac{\log \sin p}{q}$

$e^Q = \frac{\sin p}{q}$

$e^{2Q} = \frac{\sin^2 p}{q^2}$

$q^2 e^{2Q} = 1 - \cos^2 p = \cos^2 p = 1 - q^2 e^{2\theta}$

$\Rightarrow \cos p = \sqrt{1 - q^2 e^{2\theta}} \quad \Rightarrow \text{p} = \cos^{-1}\sqrt{1 - q^2 e^{2\theta}} \quad \ldots\ldots\ldots..(2)$

Sub (2) in (1)

$\Rightarrow \psi = q\cos^{-1}\left(\sqrt{1 - q^2 e^2 Q}\right) + q\text{ctp}\, p$

$= q\cos^{-1}\left(\sqrt{1 - q^2 e^{2Q}}\right) + q\frac{\cos p}{\sin p}$

$= q\cos^{-1}\left(\sqrt{1 - q^2 e^{2Q}}\right) + \frac{\cos\left(\cos 1\sqrt{1 - e^2 q^2}\right)}{e^{\text{Q}}}$

$\psi = q\cos^{-1}\left(\sqrt{1 - q^2 e^{2Q}}\right) + \frac{e^{\text{Q}}\sqrt{e^{-2Q} - q^2}}{e^{\text{Q}}}$

(I) $F_1 = q\cos^{-1}\sqrt{1 - q^2 e^{2Q}} + \sqrt{e^{-22} - q^2}$

$\frac{\partial F_1}{\partial q} = \cos^{-1}(1 - q^2 e^{2Q}) = p$

$\frac{\partial F_1}{\partial Q} = \frac{e^{-2Q}(-2)}{2\sqrt{e^{-2Q} - q^2}}$

$= -\frac{e^{-2Q}}{\sqrt{e^{-2Q} - q^2}} \cdot \frac{e^{+Q}}{e^{+Q}}$

$= -\frac{e^{-Q}}{e^2\sqrt{e^{-2Q}} - q^2}$

$= -\frac{e^{-Q}}{e^Q e^{-Q}\sqrt{1 - e^{2Q} q^2}}$

$\frac{\partial F_1}{\partial Q} = -\frac{e^{-Q}}{\sqrt{1 - q^2 e^{2Q}}}$

$(\because e^Q = \frac{\sin\phi}{q} \Rightarrow \frac{1}{e^{\text{Q}}} = \frac{q}{\sin p})$

$= \frac{-\frac{q}{\sin p}}{\cos p} = -q\cot p = -P$

$(II) F_2 = qP + Qp + q\cot p$

$qp = q\tan^{-1}(q/p)$

$QP = -p\log\left(\sqrt{q^2+p^2}\right)$ $[q\cot p = p]$

$F_2(q,p) = q\tan^{-1}\frac{q}{p} + p\left(1 - \log\sqrt{q^2+p^2}\right)$

$\frac{\partial F_2}{\partial q} = \tan^{-1}\frac{q}{p} = p$

$\frac{\partial P_2}{\partial P} = -\log\sqrt{P^2+q^2} = Q$

(III) $F_3(p,Q) = F_1 - qp = e^{-Q}\cos p$

$\frac{\partial F_3}{\partial p} = -e^{-Q}\sin p = -q$

$\frac{\partial P}{\partial Q} = -e^{-Q}\cos p = -p$

(IV) $F_4(p,p) = F_2 - qp = Qp + q\omega tp$

$= p + p\log\left(\frac{\cos p}{pq}\right)$

$\frac{\partial F_4}{\partial p} = -p\tan p = -q$

$\frac{\partial F_4}{\partial p} = \log\left(\frac{\cos p}{\cos p}\right) = Q$

5.2. Special Transformations:

Some Simple Transformations:

Let us consider the identity transformation, which is an obvious example of a canonical transformation. It is generated by a function of the form

$F_2 = \sum_{i=1}^{n} q_i P_i$

As can be confirmed by nothing that

$p_i = \frac{\partial F_2}{\partial q_i} = P_i$, $Q_i = \frac{\partial F_2}{\partial q_i} = q_i$ (i=1,2,……n)

Homogeneous Canonical Transformations:

Let us consider the case where ϕ and ψ are identically zero.

Then $\sum_{i=1}^{n} P_i \delta q_i - \sum_{i=1}^{n} P_i \delta Q_i = 0$

and the corresponding transformation is called a homogeneous canonical transformation. This transformation is also known as a Mathieu transformation (or) contact transformation.

Point Transformation:

Consider a class of Homogeneous canonical transformation for which of fill set of n-independent transformation $\Omega(q, Q, t)$ exists and equal to zero. Then,

$$\sum_{i=1}^{n} \frac{\partial \Omega q_j}{\partial q_i} \delta q_i + \sum_{i=1}^{n} \frac{\partial \Omega_j}{\partial R_i} \delta Q_i, \quad j = 1 \text{ to } m$$

are non-singular matrix.

(i.e.,) $\left|\frac{\partial \Omega_j}{\partial q_i}\right| \neq 0$

$$\left|\frac{\partial \Omega_j}{\partial Q_i}\right| \neq 0$$

we have $Q_i = f_i(q, t), i = 1$ to m.

where f is twice differential function

This equation represents a point Transformation

Momentum Transformation:

If $P_i = H_i(P, t), i = 1$ to n. This represent a point transformation in momentum space f it is called a momentum transformation.

(i.e.,) $P_i = \sum_{j=1}^{n} P_j \frac{\partial f_j}{\partial q_i}$.

5.3. Lagrange and Poisson Brackets:

Definition: Lagrange Bracket:

Let $[u, v]$ be any two variables $q_1, q_2 \cdots q_n, P_1, P_2 \dots P_n$, then

$[u, v] = \sum_{k=1}^{n} \left(\frac{\partial q_k}{\partial u}, \frac{\partial P_k}{\partial v} - \frac{\partial q_k}{\partial v} \frac{\partial P_k}{\partial u}\right)$ is called a Lagrange's Bracket.

Note:

(1) $[u, v] = -[v, u]$

(2) $[u, u] = 0 = [v, v]$.

Theorem 1:

Fundamental Lagrange bracket (or) Sufficient condition for canonical transformation

(1) $[q_i, q_j] = 0$ First Fundamental Lagrange Bracket

(2) $[P_i, P_j] = 0$ Second Fundamental Lagrange Bracket

(3) $[q_i, p_j] = \delta_{ij}$ Third Fundamental Lagrange Bracket. Where " q and p are independent".

Proof:

$$(1)[q_i, q_j] = \sum_{i=1}^{n} \left(\frac{\partial q_k}{\partial q_i} \frac{\partial p_k}{\partial q_j} - \frac{\partial q_k}{\partial q_j} \cdot \frac{\partial p_k}{\partial q_i} \right)$$

$$= 0 \qquad \left[\because \frac{\partial p_{1c}}{\partial q_j} = \frac{\partial p_k}{\partial q_i} = 0 \,, p, q \; are \; independent \right]$$

(2) $[P_i, P_j] = \sum_{i=1}^{n} \frac{\partial q_k}{\partial P_i} \cdot \frac{\partial p_k}{\partial p_j} - \frac{\partial q_{1c}}{\partial P_j} \cdot \frac{\partial P_k}{\partial P_i}$.

$$(3)[q_i, p_j] = \sum_{i=1}^{n} \left(\frac{\partial q_k}{\partial q_i} \cdot \frac{\partial p_k}{\partial p_j} - \frac{\partial q_k}{\partial p_j} \cdot \frac{\partial p_k}{\partial q_i} \right)$$

$$= \left(0 + 0 + \cdots + \frac{\partial q_i}{\partial q_i} \cdot \frac{\partial p_i}{\partial p_j} + \cdots \right) - (0 + 0 + \cdots \ldots + 0)$$

$$= 1 \cdot \frac{\partial p_i}{\partial p_j}$$

$$[q_i, p_j] = \delta_{ij}$$

Definition: Poisson Bracket:

The Poisson bracket of (u, v), where u and v are functions of $q_1, q_2 \cdots q_n, p_1, p_2, \cdots p_n$ is given by, $(u, v)_{q/p} = \sum_{i=1}^{n} \left(\frac{\partial u}{\partial q_i} \cdot \frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i} \cdot \frac{\partial v}{\partial q_i} \right)$

Result:

(1) $(u, v) = -(v, u)$

(2) $(u, u) = (v, v) = 0$.

Proof:

Prove that (1) $(q_i, q_j) = 0$

(2) $(P_i, P_j) = 0$

(3) $(q_i, p_j) = \delta_{ij}$ for p, q are independent.

Derive Jacobian Identity:

Let u, v, w are functions of $q_1, q_2 \cdots - q_n$, $P_1, P_2 \ldots P_n$. Then prove that

$(u, (v, w)) + (v, (w, u)) + (w, (u, v)) = 0$

Proof:

Let, $u = u(q_i, p_i)$

$V = v(q_i, p_i)$

$w = w(q_i, p_i)$

∴ By poisson Bracket,

$$(v, w)_{(q,p)} = \sum_{i=1}^{n} \left(\frac{\partial v}{\partial q_i} \cdot \frac{\partial w}{\partial p_i} - \frac{\partial v}{\partial p_i}\frac{\partial w}{\partial q_i}\right)$$

$$(u, (v, w)) = \sum_{k=1}^{n} \left[\frac{\partial u}{\partial q_k} \cdot \frac{\partial (v, w)}{\partial p_k} - \frac{\partial u}{\partial p_k}\frac{\partial (v, w)}{\partial q_k}\right]$$

$$= \sum_{k=1}^{n} \frac{\partial u}{\partial q_k} \sum_{k=1}^{n} \left[\left(\frac{\partial^2 u}{\partial p_i\, \partial q_k} \cdot \frac{\partial w}{\partial p_i} + \frac{\partial v}{\partial q_i}\frac{\partial^2 w}{\partial p_i\, \partial p_k}\right) - \left(\frac{\partial^2 v}{\partial p_k\, \partial p_i} \cdot \frac{\partial w}{\partial q_i} + \frac{\partial v}{\partial p_i} \cdot \frac{\partial^2 w}{\partial p_k\, \partial q_i}\right)\right]$$

$$- \sum_{i=1}^{n} \left[\frac{\partial u}{\partial p_k} \sum_{i=1}^{n} \left[\left(\frac{\partial^2 v}{\partial q_i\, \partial q_k}\frac{\partial w}{\partial p_i} + \frac{\partial v}{\partial q_i} \cdot \frac{\partial^2 w}{\partial p_i\, \partial q_k}\right) - \frac{\partial^2 v}{\partial R_i\, \partial q_k}\frac{\partial w}{\partial q_i} + \frac{\partial v}{\partial p_i} \cdot \frac{\partial^2 w}{\partial q_k}\right)\right.$$

$$= \sum_{i=1}^{n} \sum_{i=1}^{n} \left(\frac{\partial u}{\partial q_i} - \frac{\partial^2 v}{\partial p_i\, \partial q_i} \cdot \frac{\partial w}{\partial p_i} + \frac{\partial u}{\partial q_i} \cdot \frac{\partial v}{\partial q_i} - \frac{\partial u}{\partial q_i}\frac{\partial^2 u}{\partial p_i} \cdot \frac{\partial w}{\partial q_i} - \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i}\right]$$

Theorem 2:

Obtain relation between Poisson Bracket and Lagrangian bracket.

Let u_l , $l = 1,2 \ldots 2n$ be an $2n$ idependent functions of $2n$ variables $q_1, q_2 \cdots q_n$, $p, p_2 \cdots p_n \cdots$ then $\sum_{l=1}^{2n} [u_l, u_i](u_l, u_j) = \delta_{ij}$ and $\sum_{l=1}^{2n} [u_l, u_i](u_l, u_i) = 1$

Proof:

LHS

$$\sum_{l=1}^{2n} [u_l, u_i](u_l, u_j) = \sum_{i=1}^{2n} \sum_{k=1}^{n} \sum_{in=1}^{n} \left(\frac{\partial q_k}{\partial u_l} \cdot \frac{\partial p_k}{\partial u_i} - \frac{\partial q_k}{\partial u_i} \cdot \frac{\partial p_{ik}}{\partial u_l}\right) . \left(\frac{\partial u_l}{\partial q_m} \cdot \frac{\partial u_j}{\partial p_m} - \frac{\partial u_l}{\partial p_m}\frac{\partial u_j}{\partial q_m}\right)$$

First term:

$$\sum_{k,m=1}^{n}\left(\frac{\partial p_k}{\partial u_i}\cdot\frac{\partial u_i}{\partial p_m}\right)_{m=1}^{2n}\sum_{l=1}^{\partial q_l}\cdot\frac{\partial q_l}{\partial q_m}$$

$$=\sum_{k,m=1}^{n}\left(\frac{\partial p_k}{\partial u_i}\cdot\frac{\partial u_j}{\partial p_m}\right)\frac{\partial q_k}{\partial q_m}$$

$$=\sum_{k,m=1}^{n}\frac{\partial p_k}{\partial u_i}\cdot\frac{\partial u_j}{\partial p_m}\delta k_m$$

$$=\sum_{k=1}^{n}\frac{\partial u_j}{\partial p_k}\cdot\frac{\partial p_k}{\partial u_i}\cdot 1\ \ldots\ldots\ldots..(1)\quad [\because \delta k_{km}=1, \text{if } k=m]$$

Second term:

$$-\sum_{k_1m=1}^{n}\left(\frac{\partial p_k}{\partial u_i}\cdot\frac{\partial u_j}{\partial q_m}\right)\sum_{l=1}^{2n}\frac{\partial q_k}{\partial u_l}\cdot\frac{\partial u_l}{\partial p_m}$$

$$\sum_{k,m=1}^{n}\left(\frac{\partial p_k}{\partial u_i}\cdot\frac{\partial u_j}{\partial q_m}\right)\frac{\partial q_k}{\partial p_m}=0\quad [\because\ q_k\ are\ independent\ p_m]$$

Third term:

$$\sum_{k,m=1}^{n}\left(\frac{\partial p_k}{\partial u_i}\cdot\frac{\partial u_j}{\partial p_m}\right)\sum_{l=1}^{2n}\frac{\partial p_k}{\partial u_l}\frac{\partial u_l}{\partial q_m}$$

$$=-\sum_{k_1m=1}^{n}\left(\frac{\partial q_k}{\partial u_i}\cdot\frac{\partial u_j}{\partial p_m}\right)\frac{\partial p_k}{\partial q_m}$$

=0 $\quad [\because\ p_k\ are\ independent\ q_m]$

Fourth term:

$$\sum_{k,m=1}^{n}\left(\frac{\partial q_k}{\partial u_i}\cdot\frac{\partial u_j}{\partial q_m}\right)\sum_{l=1}^{2n}\frac{\partial p_k}{\partial u_l}\frac{\partial u_l}{\partial p_m}$$

$$=\sum_{k_1m=1}^{n}\left(\frac{\partial q_k}{\partial u_i}\cdot\frac{\partial u_j}{\partial q_m}\right)\frac{\partial p_k}{\partial q_m}$$

$$=\sum_{k_1m=1}^{n}\frac{\partial q_k}{\partial u_i}\cdot\frac{\partial u_j}{\partial q_m}\delta k_{km}$$

$$=\textstyle\sum_{k_1m=1}^{n}\frac{\partial q_k}{\partial u_i}\cdot\frac{\partial u_j}{\partial q_m}\ (1)\quad [\because \delta k_{km}=1, \text{if } k=m]$$

Add:

L.H.S: $\sum_{l=1}^{2n} [u_l, u_i](u_l, u_i)$

$$= \sum_{k_1 m=1}^{n} \left(\frac{\partial u_j}{\partial p_k} \cdot \frac{\partial p_k}{\partial u_i} + \frac{\partial q_k}{\partial u_i} \frac{\partial u_j}{\partial q_k} \right)$$

$= \frac{\partial u_j}{\partial u_i} \quad [\because u = u(q_i, p_i)]$

$= \delta_{ij}$

Hence $\sum_{l=1}^{2n} [u_l, u_i](u_l, u_j) = \delta_{ij} = 1 \quad if\ i = j.$

Example 1:

Derive the differential equation of constant of motion of a system.

Proof:

consider the function $f(q, p, t)$.

$\frac{df}{dt} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} \cdot \dot{q}_i + \frac{\partial f}{\partial p_i} \cdot \dot{p}_i \right) + \frac{\partial f}{\partial t}$

$\quad = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} \cdot \frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}, \frac{\partial H}{\partial q_i} \right) + \frac{\partial f}{\partial t}$ [by Hamilton canonical equation].

$\frac{df}{dt} = (f, H) + \frac{\partial f}{\partial t}$

It f is a constant. then $(f, H) + \frac{\partial f}{\partial t} = 0$

is called differentiation equation of constant of motion of a system.

Theorem 3:

State and prove Poisson's theorem:

If $u(q, p)$ and $v(q, p)$ are integrals of a system with Hamiltonian $H(q, p, t)$. Then poisson's Integrals of (u, v) is also an integral of motion.

Proof:

Given that u of & v are constants of motion, then we have,

$$(u, H) + \frac{\partial u}{\partial t} = 0 \quad \ldots\ldots\ldots..(1) \qquad \frac{\partial u}{\partial t} = -(u, H)$$

$$(v, H) + \frac{\partial v}{\partial t} = 0 \quad \ldots\ldots\ldots.(2) \qquad \frac{\partial v}{\partial t} = -(v, H)$$

and $\frac{d}{dt}(u, v) = ((u, v), H) + \frac{\partial (u,v)}{\partial t}$

$= ((u,v),H) + \left(\frac{\partial u}{\partial t},v\right) + \left(u,\frac{\partial v}{\partial t}\right).$ $\quad by\ (1)\ \&\ (2)$

$= \big((u,v),H\big) + (-(u,H),v) + \big(u,-(v,H)\big)$

$= ((u,v),H) + ((H,u),v) + ((v,H),u)$

$= ((u,v),H) + ((v,H),u) + ((H,u),v))$

=0

i.e., (u, v) = 0

i.e., $\frac{d}{dt}$ (u, v)=a constant

i.e., The Poisson bracket of two constants of motion is also a constant of motion.

Theorem 4:

Derive Hamilton equation of motion in terms of Poisson Bracket.

(or) prove that $\dot{q}_i = (q_i, \mathrm{H}),\ \ \dot{P}_i = (P_i, H)$

Proof:

Consider the holonomic system with canonical co-ordinates (q, p) and Hamiltonian H

$$(q_i,\mathrm{H}) = \sum_{i=1}^{n}\left(\frac{\partial q_i}{\partial q_i}\cdot\frac{\partial H}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\frac{\partial \mathrm{H}}{\partial q_k}\right)$$

$$= \sum_{i=1}^{n}\left(\frac{\partial q_i}{\partial q_k}\cdot \dot{q}_k + 0.\dot{P}_k\right)$$

$= \sum_{k=1}^{n}\ \delta_{ik}\cdot\dot{q}_k$

$(q_i,\mathrm{H}) = \dot{q}_i$

$$(P_i,H) = \sum_{i=1}^{n}\left(\frac{\partial p_i}{\partial q_k}\cdot\frac{\partial H}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\frac{\partial \mathrm{H}}{\partial q_k}\right)$$

$$= \sum_{i=1}^{n}\ (0.\dot{q}_k + \delta_{ik}\,\dot{P}_k)$$

$$= 0 + \sum_{k=1}^{n}\ \delta_{ik}\cdot\dot{P}_k$$

$(P_i,H) = \dot{P}_i$

hence Hamilton canonical equation of motion in terms of Poisson Brackets. If

(1) $\dot{P}_i = (P_i, H)$

(2) $\dot{q}_i = (q_i, \mathrm{H})$

Exercises:

1. For a certain canonical transform it is known that $Q = \sqrt{q^2 + p^2}$,

   $\psi = \frac{1}{2}(q^2 + p^2)tan^{-1}\frac{q}{p} + \frac{1}{2}qp$ Find P(q,p) and ϕ(q,Q)

2. A particle of mass m moves in the xy plane under the action of a potential function V=ky. For a homogeneous point transform $Q_1 = xy, Q_2 = \frac{1}{2}(x^2 - y^2)$. Find the expression for P_1 and P_2 and the generating function F_2(q, p). What is the new Hamiltonian function K(Q, P)

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